Serre curves in one parameter families:

Joint work w/ A. Cojocaru and D. Grant.

§1 Introduction:

K = # field, E/K an elliptic curve, $G_K = \mathrm{Gal}(\bar{K}/K)$,
$E[n]$ = n-torsion of E, $E_{tors} = \bigcup_{n \geq 1} E[n]$.
As $E[n]$ is a Galois module, one has
$$\varphi_{E,n} : G_K \longrightarrow \mathrm{Aut}(E[n]) \simeq GL_2(\mathbb{Z}/n\mathbb{Z}).$$

Note: $\varphi_{E,n}$ may not be surjective. For example, $X_0(11)$ is given by
$$y^2 + y = x^3 - x^2 - 10x - 20.$$
This has a rational 5-torsion pt. $(5,5) = P \in E[5](\mathbb{Q})$.
Let $\{P, Q\}$ be a basis of $E[5]$. But then
$$\varphi_{X_0(11),5}(G_{\mathbb{Q}}) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\},$$

so is not surjective.

One also has
$$\varphi_E : G_K \longrightarrow \mathrm{Aut}(E_{tors}) \simeq GL_2(\hat{\mathbb{Z}}) \simeq \prod_p GL_2(\mathbb{Z}_p).$$

Thm (Serre): If E/K has no CM then
$$[GL_2(\hat{\mathbb{Z}}) : \varphi_E(G_K)] < \infty.$$
(Equivalently, $\exists\, C_{E,K} > 0$ s.t. $\forall$ prime $\ell \geq C_{E,K}$ the $\varphi_{E,\ell}(G_K) = GL_2(\mathbb{Z}/\ell\mathbb{Z})$.)

<u>Question (Serre):</u> Can one take $C_{E,K} = C_K$ independent of E?

Even when $K = \mathbb{Q}$, this question is still open. It is generally believed that $C_{\mathbb{Q}} = 41$ should work.

This question has applications to solutions to Diophantine equations.

For $H \leq GL_2(\mathbb{Z}/n\mathbb{Z})$, H acts on the modular curve $X(n)$. We set

$$X_H := {}_H\backslash X(n).$$

Assume $-I \in H$, $\det: H \longrightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ is surjective.

$$X_H(\mathbb{Q}) \longleftrightarrow \{\text{cusps}\} \sqcup \{E/\mathbb{Q} \text{ s.t. } \varphi_{E,n}(G_{\mathbb{Q}}) \subseteq H\}.$$

Known results: $(/\mathbb{Q})$

	<u>Borel</u>	<u>Normalizer of split Cartan</u>
H	$\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\} \sqcup \left\{\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right\}$
	Mazur ✓	Bilu - Parent ✓
	<u>Non-split Cartan</u>	<u>Exceptional groups</u>
	Open	Serre ✓

<u>Def:</u> n is <u>exceptional</u> for E if $\varphi_{E,n}(G_K) \subsetneq GL_2(\mathbb{Z}/n\mathbb{Z})$.

§2. Average Result:

Consider elliptic curves $E: y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$.

$H(E) = \max\{|a|^3, |b|^2\}$

$\mathcal{F}(x) = \{E : H(E) \le X^6\}$

Thm (Duke '97):

$$\frac{\#\{E \in \mathcal{F}(x) : E \text{ has at least one exceptional prime}\}}{\# \mathcal{F}(x)} \ll \frac{X^4 (\log X)^\gamma}{X^5}$$

size of box

$\gamma > 0$ an explicit constant.

Cor: For almost all $E \in \mathcal{F}(x)$, one may take $C_{E,\mathbb{Q}} = 1$.

Thm (Grant '00):

$$\#\{E \in \mathcal{F}(x) : E \text{ has at least one excep. prime}\} \sim C \cdot X^3.$$

Main term comes from E ~~exceptional~~ @ $\ell = 2$ and 3.

Can φ_E be surjective? The answer is no if $K = \mathbb{Q}$.

$$\begin{tikzcd}
 & \mathbb{Q}(E[\mathrm{lcm}(2,n)]) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(E[2]) \arrow[d, dash] \arrow[dd, bend right=60, dash, "GL_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3"'] & & \mathbb{Q}(E[n]) \\
\mathbb{Q}(\sqrt{\Delta_E}) \arrow[d, equal] & & \mathbb{Q}(\zeta_n) \arrow[u] \arrow[ll, dash] \\
\mathbb{Q} & &
\end{tikzcd}$$

(For $K \neq \mathbb{Q}$ see A. Greicius' PhD Thesis. Can be surj if $K(\sqrt{\Delta_E}) \not\subseteq K^{cyc}$.)

$\Rightarrow$

$$\varphi_{E,2n}(G_{\mathbb{Q}}) \subseteq \{g \in GL_2(\mathbb{Z}/2n\mathbb{Z}) : \underset{\text{sign char}}{\varepsilon}(g \bmod 2) = \left(\frac{\Delta_i}{\det g}\right)\}$$

Call $E/\mathbb{Q}$ a Serre curve if $[GL_2(\hat{\mathbb{Z}}) : \varphi_E(G_{\mathbb{Q}})] = 2$,
i.e., if φ_E is as surjective as possible.

Thm (J. '05):

$$\#\{E \in \mathcal{F}(X) : E \text{ is not a Serre curve}\} \ll X^4/(\log X)^\gamma$$

One can see Lang-Trotter for examples of Serre curves.
One gets the same asymptotic as Grant's if one considers non-Serre curves.

Thinner Asym: Consider

$$(*) \quad \mathbb{E} : y^2 = x^3 + A(t)x + B(t) \qquad A(t), B(t) \in \mathbb{Z}[t]$$
$$j_{\mathbb{E}}(t) \notin \mathbb{Q}.$$

height

$$\mathcal{F}_{\mathbb{E}}(T) = \{t_0 \in \mathbb{Q} : H(t_0) \le T, \mathbb{E}_{t_0} \text{ is an elliptic curve}\}. \asymp T^2$$

Thm (Cojocaru-Hall '05):

$$\#\{t_0 \in \mathcal{F}(T) : \mathbb{E}_{t_0} \text{ has some exceptional } \ell \ge 17\} \ll T^{3/2}(\log T)^\gamma.$$

§3 Our theorem:

$$\mathbb{X} = \{X_0(2), X_{A_3}(2), X_0(3), X'(4), X_0(5), \ldots\}$$

a finite list.

These are the genus 0 curves among those which parameterize non-Serre curves.

$$\mathcal{Y}_{\mathbb{E}} = \{ x \in \mathbb{X} : \quad X \times_{\mathbb{P}^1(j)} \mathbb{P}^1(t) \quad \text{has genus } 0 \}$$

$$\begin{tikzcd}
X \times_{\mathbb{P}^1(j)} \mathbb{P}^1(t) \arrow[r] \arrow[d] & \mathbb{P}^1(t) \arrow[d, "j_{\mathbb{E}}"] \\
X \arrow[r, "j_X"'] & \mathbb{P}^1(j)
\end{tikzcd}$$

$$\mathcal{E}_{\text{non-Serre}}(T) = \{ t_0 \in \mathcal{F}(T) : \mathbb{E}_{t_0} \text{ is a non-Serre curve} \}$$

$$d = d_{\mathbb{E}} = \min \{ \deg j_X : X \in \mathcal{Y}_{\mathbb{E}} \}, \quad d \geq 2.$$

<u>Thm (Cojocaru-Grant-J.):</u> Given $\mathbb{E}$ as in (*) satisfying

$$\varphi_{\mathbb{E},n}(G_{\mathbb{Q}(t)}) = GL_2(\mathbb{Z}/n\mathbb{Z}) \quad \text{for each } n \geq 1,$$

then given $\varepsilon > 0$

$$\# \mathcal{E}_{\text{non-Serre}}(T) = \begin{cases} O_{\mathbb{E},\varepsilon}(T^{1+\varepsilon}) & \text{if } \left(j_{\mathbb{E}}(T) = P\left(\frac{at+b}{ct+d}\right), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}), \ \cancel{\text{nonsingular and}} \ P(x) \in \mathbb{Q}[x] \right) ** \\ \sim C_{\mathbb{E}} T^{2/d_{\mathbb{E}}} & \text{if not } ** \text{ and } \mathcal{Y}_{\mathbb{E}} \neq \emptyset. \\ O_{\mathbb{E},\varepsilon}(T^{\varepsilon}) & \text{o/w.} \end{cases}$$

<u>Sketch of proof:</u>

① $\mathcal{E}_{\text{non-Serre}}(T) = \mathcal{E}_{36}^{\text{non-Serre}}(T) \cup \left(\bigcup_{\ell \text{ prime}} \mathcal{E}_\ell(T) \right)$

where

$$\mathcal{E}_n(T) = \{ t_0 \in \mathcal{F}(T) : \mathbb{E}_{t_0} \text{ is exceptional at } n \}$$

This is purely a group theory result.

② $\bigcup_{\ell} \mathcal{E}_\ell(T) \subseteq \left(\bigcup_{\ell \leq r} \mathcal{E}_\ell(T) \right) \cup \mathcal{E}^{\wedge}_{\text{lhr}}(T)$, where

$r \geq 17$, and

$$\mathcal{E}^r_{int}(T) = \left\{ \frac{R_0}{S_0} \in \mathcal{F}(T) : d_r(G(R_0,S_0)) \mid F(R_0,S_0) \right\}$$

where

$$j_E\left(\frac{R}{S}\right) = \frac{F(R,S)}{G(R,S)} \in \mathbb{Q}(R,S)$$ homog, lowest terms

$d_r(n)$ $\qquad n = 2^\alpha 3^\beta c_r(n) d_r(n)$

$c_r(n)$: r-full, coprime to 6 $\qquad d_r(n)$: r-free.

③ $\mathcal{E}^r_{int}(T) \ll T^{1+\varepsilon+n/r}$ if ** occurs.

$\qquad T^{\varepsilon+n/r}$ if not $\qquad n = \deg$ of j_E

④ $\#\left(\mathcal{E}^{\text{non-serre}}_{36}(T) \cup \bigcup_{\ell \le r} \mathcal{E}_\ell(T)\right) \sim \begin{cases} c\,(T^{?/6}) & \text{if } \mathcal{Y}_E \ne \phi. \\ O_\varepsilon(T^\varepsilon) & \text{else} \end{cases}$

⑤ Choose r large enough.