Gross
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pg 1

Self-dual and conjugate-dual Galois representations:

k = local field

W = Weil group of k (can be done for Weil-Deligne group as well)

<u>Recall:</u> $W \longrightarrow \mathrm{Gal}(k^s/k)$

$k^\times \longrightarrow W^{ab}$ via LCFT.

We are interested in $W \longrightarrow GL(M)$ $\mathbb{C}$-rep. of Weil rep.

$M^\vee = \mathrm{Hom}(M, \mathbb{C})$

$\det M = \Lambda^{top} M : k^\times \longrightarrow \mathbb{C}^\times$ cont. homom.

$\varepsilon(M, \Psi, dx, 1/2) = \varepsilon(M, \Psi) \in \mathbb{C}^\times$ (local ~~root~~ numbers.)

$\Psi : k^\times \rightarrow S^1$ nontrivial additive char.

dx = Haar measure on k^+. We choose this to be self-dual wrt Fourier transform w/ Ψ.

$\Psi_a = \Psi(ax)$, $a \in k^\times$.

$\varepsilon(M, \Psi_a) = \varepsilon(M, \Psi) \det M(a)$. Note this is indep of Ψ if $\det M = 1$.

$\varepsilon(M, \Psi)\varepsilon(M^\vee, \Psi^{-1}) = 1$. (i.e. $\varepsilon(M,\Psi)\varepsilon(M^\vee, \Psi) = \det M(-1)$.)

Self-dual means $M \simeq M^\vee$ as reps of W. (i.e. we have $\det M \simeq \det M^\vee = (\det M)^\vee = (\det M)^{-1} \Rightarrow (\det M)^2 = 1$)

We also get $\varepsilon(M, \Psi)^2 = \det M(-1) = \pm 1$.

If $\det M = 1$, then $\varepsilon(M)^2 = 1$. Thus, $\varepsilon(M) = \pm 1$ for a self-dual representation with det 1.

$M \xrightarrow{f} M^\vee \rightsquigarrow B : M \times M \longrightarrow \mathbb{C}$ non-deg. bilinear form.

$M \xrightarrow{{}^t f} M^\vee$ $\qquad B(wm, wn) = B(m, n) \quad \forall w \in W$.

We say the rep. is orthogonal if $B(m,n) = B(n,m)$. $(f = {}^t f)$

We say the rep. is symplectic if $B(m,n) = -B(n,m)$ $(f = -{}^t f)$.

For symplectic, $\varepsilon(M) = \pm 1$. This is very mysterious. Many examples have been computed, but no general framework.

Assume for simplicity that M is orth. of $\det M = 1$.

$$\begin{tikzcd} W \arrow[r] \arrow[dr, dotted] & SO(M) \subset GL(M) \\ & Spin(M) \arrow[u] \end{tikzcd} \qquad \varepsilon(M) = \pm 1$$

Then Deligne shows that $\varepsilon(M) = 1$ iff (if char $k \neq 2$) $W \to SO(M)$ lifts to $W \to Spin(M)$. (if char $k = 2$, then $\varepsilon(M) = 1$ always)

M orth., $\dim M = 2$, $\det M = 1$. Then $M = \chi + \chi^{-1}$.

$$\varepsilon(M, \psi) = \varepsilon(\chi, \psi)\, \varepsilon(\chi^{-1}, \psi)$$

$$\Rightarrow \quad \varepsilon(M) = \chi(-1). \qquad (-1 \in k^* \text{ is unique elt of order } 2)$$

So the orthogonal case is pretty much known, the symplectic case is very mysterious.

<u>We now switch to conjugate-dual:</u>

$$\begin{tikzcd} K \arrow[d, dash, "\text{quad ext}", "\langle 1,\sigma\rangle"'] \\ k \end{tikzcd} \qquad W_K \underset{2}{\triangleleft} W \qquad W/W_K = \langle 1, \sigma \rangle.$$

Choose rep. $s \in W$, $s^2 \in W_K$ and $s \mapsto \sigma \in W/W_K$.

$W_K \longrightarrow GL(M)$

$w_K(m) = s w_K s^{-1}(m)$: M^s is this rep.

Choosing a different s gives an isomorphic rep.

M^σ is the isom. class of M^s.

means same isom class.

Conjugate dual means $\boxed{M^\sigma \simeq M^\vee}$:

$M^s \to M^\vee$ $\quad B: M\times M \longrightarrow \mathbb{C}$ non-deg. bilinear form

$$B(wm, sws^{-1}n) = B(m,n).$$

Choose $\psi: K^+ \to S^1$ satisfying $\psi^\sigma = \psi^{-1}$ where $\psi^\sigma(x) = \psi(x^\sigma)$.

All other ψ's of this form are of the form ψ_a with $a \in k^\times$.

$$1 = \varepsilon(M,\psi)\,\varepsilon(M^\vee,\psi^{-1})$$
$$= \varepsilon(M,\psi)\,\varepsilon(M^\sigma,\psi^\sigma)$$
$$= \varepsilon(M,\psi)^2$$

$\Rightarrow \quad \varepsilon(M,\psi) = \pm 1.$

Now $\varepsilon(M,\psi_a) = \varepsilon(M,\psi)\det M(a)$

$$\det M : \underset{\substack{\cup \\ a\in k^\times}}{K^\times} \longrightarrow \mathbb{C}^\times$$

So we need to look at the det. char. for conj. dual reps.

$\det M : K^\times \longrightarrow \mathbb{C}^\times. \qquad (\det M)^{1+\sigma} = 1$

$$\underset{\substack{\uparrow \\ CFT}}{k^\times \underset{2}{\supset}} \ \mathbb{N}_m K^\times \longrightarrow 1$$

There are two possibilities:

- $\det M = 1$ on $k^\times$ $\quad \varepsilon(M)^2 = 1$
- $\det M = \alpha_{K/k}$ on $k^\times$ $\quad \varepsilon(M,\psi)^2 = 1$ depends on ψ.

For conj. dual:

sign of M

- conjugate-orthogonal if $B(m,n) = +B(m, s^2 n)$
- conj-symplectic if $B(m,n) = -B(m, s^2 n)$

Dim M = 1:

$$K^\times / Nm\, K^\times \longrightarrow \mathbb{C}^\times$$

rest. to $k^\times$ $= 1 \Longleftrightarrow$ conj. orth.

$= \alpha_{K/k} \Longleftrightarrow$ conj. sym

$$\begin{array}{ccc} & W(K/k) & \longrightarrow Gal(K/k) \\ & \nearrow & \parallel \\ k^\times & & Nm_{D^\times}(K^\times). \end{array}$$

M conj. dual with sign c

det M conj. dual with sign $c^{\det M}$

M conj. dual $\Rightarrow \varepsilon(M) = \pm 1$ indep. of ψ.

Thm: clm fact, $\varepsilon(M) = +1$ in this case.

For dim M = 1, this is a result of Fröhlich-Queyrut.

$G = GL(n)$ over k

Langlands parameters of irred. reps are just rep. M of W of $\dim M = n$.

$G = SO(2n+1)$, ${}^LG = Sp_{2n}(\mathbb{C})$

parameter $\longleftrightarrow$ symplectic M of dim $2n$.

$G = Sp_{2n}/k$

parameter $\longleftrightarrow$ orth. M of dim $2n+1$, $\det M = 1$.

$G = SO(2n)/k = SO(V)$

parameter $\longleftrightarrow$ orth. M of dim $2n$, $\det M = \operatorname{disc} V$

$G = U(n, K/k)$ $\quad {}^LG = GL(n) \cdot \mathrm{Gal}(K/k)$

$\subset GL_n(K)$

$$\begin{array}{ccc} W_k & \longrightarrow & {}^LG \\ \triangledown & & \triangledown \\ W_K & \longrightarrow & GL(n) = GL(M) \end{array}$$

A parameter for $U(n)$ $\longleftrightarrow$ M conjugate dual rep. of dim n w/ sign $c = (-1)^{n-1}$