

Self-dual and conjugate-dual Galois representations:

k = local field

W = Weil group of k (can be done for Weil-Deligne group as well)

Recall: $W \rightarrow \text{Gal}(k^s/k)$
 $k^\times \rightarrow W^{\text{ab}}$ via LCFT.

We are interested in $W \rightarrow \text{GL}(M)$ \mathbb{C} -rep. of Weil rep.

$M^\vee = \text{Hom}(M, \mathbb{C})$

$\det M = \bigwedge^{\text{top}} M : k^\times \rightarrow \mathbb{C}^\times$ cont. homom.

$\varepsilon(M, \psi, dx, \frac{1}{2}) = \varepsilon(M, \psi) \in \mathbb{C}^\times$ (local ~~number~~ number.)

$\psi: k^\times \rightarrow S^1$ nontrivial additive char.

dx = Haar measure on k^\times . We choose this to be self-dual wrt Fourier transform w/ ψ .

$\psi_a = \psi(ax)$, $a \in k^\times$.

$\varepsilon(M, \psi_a) = \varepsilon(M, \psi) \det M(a)$. Note this is indep of

ψ if $\det M = 1$.

$\varepsilon(M, \psi) \varepsilon(M^\vee, \psi^{-1}) = 1$. (i.e., $\varepsilon(M, \psi) \varepsilon(M^\vee, \psi) = \det M(-1)$.)

Self-dual means $M \cong M^\vee$ as reps of W . (i.e., we have

$$\det M \cong \det M^\vee = (\det M)^\vee = (\det M)^{-1} \Rightarrow (\det M)^2 = 1$$

We also get $\varepsilon(M, \psi)^2 = \det M(-1) = \pm 1$.

cf $\det M = \pm 1$, then $\varepsilon(M)^2 = 1$. Thus, $\varepsilon(M) = \pm 1$ for a self-dual representation with $\det 1$.

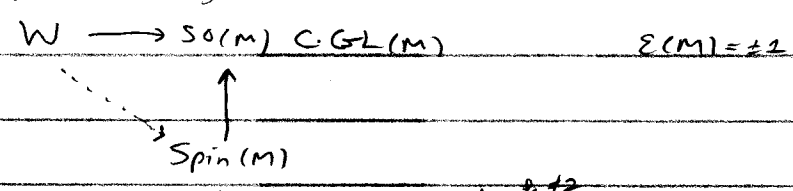
$$\begin{array}{ccc} M \xrightarrow{f} M^\vee & \rightsquigarrow & B: M \times M \rightarrow \mathbb{C} \quad \text{non-deg. bilinear form} \\ M \xrightarrow{f^\vee} M^\vee & & B(wm, wn) = B(m, n) \quad \forall w \in W. \end{array}$$

We say the rep. is orthogonal if $B(m,n) = B(n,m)$. ($f = \bar{f}$)

We say the rep. is symplectic if $B(m,n) = -B(n,m)$. ($f = -\bar{f}$)

For symplectic, $\epsilon(M) = \pm 1$. This is very mysterious. Many examples have been computed, but no general framework.

Assume for simplicity that M is orth. of $\det M = 1$.



Then Deligne shows that $\epsilon(M) = 1$ iff $W \rightarrow SO(M)$ lifts to $W \rightarrow Spin(M)$. (if $\dim k = 2$, then $\epsilon(M) = 1$ always)

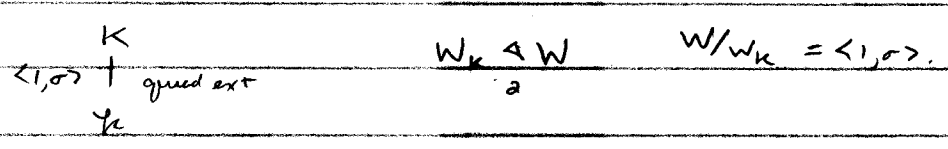
M orth. $\dim M = 2$, $\det M = 1$. Then $M = \chi + \chi^{-1}$.

$$\epsilon(M, \psi) = \epsilon(\chi, \psi) \epsilon(\chi^{-1}, \psi)$$

$$\Rightarrow \epsilon(M) = \chi(-1). \quad (-1 \in k^\times \text{ is unique elt of order 2})$$

As the orthogonal case is pretty much known, the symplectic case is very mysterious.

We now switch to conjugate-dual:



Choose rep. $s \in W$, $s^2 \in W_k$ and $s \mapsto \sigma \in W/W_k$.

$$W_K \rightarrow GL(M)$$

$$W_K(m) = S W_K S^{-1}(m) : M^S \text{ is the rep.}$$

Choosing a different S gives an isomorphic rep.

M^σ is the sim. class of M^S .

Conjugate dual means $M^\sigma \cong M^\vee$:
 means same isom. class.

$$M^S \rightarrow M^\vee \quad B: M \times M \rightarrow \mathbb{C} \text{ non-deg. bilinear form}$$

$$B(wm, S W S^{-1}m) = B(m, m).$$

Choose $\psi: K^x \rightarrow S^\pm$ satisfying $\psi^\sigma = \psi^{-1}$ where

$$\psi^\sigma(x) = \psi(x^\sigma).$$

All other ψ 's of this form are of the form ψ_a with $a \in K^x$.

$$1 = \varepsilon(M, \psi) \varepsilon(M^\vee, \psi^{-1})$$

$$= \varepsilon(M, \psi) \varepsilon(M^\sigma, \psi^\sigma)$$

$$= \varepsilon(M, \psi)^2$$

$$\Rightarrow \varepsilon(M, \psi) = \pm 1.$$

$$\text{Now } \varepsilon(M, \psi_a) = \varepsilon(M, \psi) \det M(a)$$

$$\det M: K^x \rightarrow \mathbb{C}^x$$

\cup
 $a \in K^x$

So we need to look at the det. class for conj. dual reps.

$$\det M: K^x \rightarrow \mathbb{C}^x \quad (\det M)^{120} = 1$$

$$\begin{array}{ccc} & \cup & \cup \\ \mathbb{C}^x & \xrightarrow{\det} & \mathbb{C}^x \\ & \cap & \cap \\ & \text{CFT} & \end{array}$$

There are two possibilities:

- $\det M = 1$ on k^x $\varepsilon(M)^2 = 1$
- $\det M = \alpha_{K/k}$ on k^x $\varepsilon(M, \psi)^2 = 1$ depends on ψ .

For conj. dual:

- ^{conjugate-}orthogonal if $B(m, n) = \overset{\text{sign of } M}{+} B(m, s^2 n)$
- conj-symplectic if $B(m, n) = - B(m, s^2 n)$

$\dim M = 1$:

$$K^x / N_m K^x \longrightarrow \mathbb{C}^x$$

$$\text{rest. to } k^x = 1 \iff \text{conj. orth.}$$

$$= \alpha_{K/k} \iff \text{conj. sym.}$$

$$\begin{array}{ccc} & W(K/k) & \longrightarrow \text{Gal}(K/k) \\ & \nearrow & \\ k^x & \text{Nm}_{D^x}(K^x) & \end{array}$$

M conj. dual with sign c

$\det M$ conj. dual with sign $c^{\det M}$

M conj. dual $\Rightarrow \varepsilon(M) = \pm 1$ indep. of ψ .

Thm: class fact. $\varepsilon(M) = +1$ in this case.

For $\dim M = 1$, this is a result of Frobenius-Orezt.

$$G = GL(n) \text{ over } k$$

Langlands parameters of irred. reps are just rep. M of W
of $\dim M = n$.

$$G = SO(2n+1), \quad {}^L G = Sp_{2n}(\mathbb{C})$$

parameters \leftrightarrow symplectic M of $\dim 2n$.

$$G = Sp_{2n}/k$$

parameters \leftrightarrow orth. M of $\dim 2n+2$, $\det M = 1$.

$$G = SO(2n)/k = SO(V)$$

parameters \leftrightarrow orth. M of $\dim 2n$, $\det M = \det V$

$$G = U(n, K/k) \quad {}^L G = GL(n) \cdot Gal(K/k)$$

$$GL_n(K)$$

$$W_K \longrightarrow {}^L G$$

$$\triangleright \quad \triangleright$$

$$W_K \longrightarrow GL(n) = GL(M)$$

A parameter for $U(n) \leftrightarrow M$ conjugate dual rep. of $\dim n$

$$\text{w/ sign } \epsilon = (-1)^{n-1}$$