Iwasawa Theory and Projective Modules:

Notation:

$$\begin{tikzcd}
 & K_\infty \arrow[dl, dash, "\Gamma"'] \arrow[dr, dash, "\Delta"] & \\
K \arrow[dr, dash, "\Delta"'] & & F_\infty \arrow[dl, dash, "\Gamma"] \\
 & F &
\end{tikzcd}$$

F_∞ cyclo $\mathbb{Z}_p$-ext

Assume (though one can avoid this) $K \cap F_\infty = F$.

$\Sigma_{K/F} = \{ v \mid v \nmid p\infty,\ e_v(K/F) \text{ is divisible by } p \}$.

E = elliptic curve defined over F with good ordinary red. for all $v \mid p$.

Conjecture (Mazur): $\mathrm{Sel}_E(K_\infty)_p$ is a cotorsion-module over $\Lambda = \mathbb{Z}_p[[\Gamma]]$. (this means $X_E(K_\infty) = \mathrm{Hom}(\mathrm{Sel}_E(K_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$ is a f.g. torsion Λ-module, i.e., $X_E(K_\infty) \simeq \mathbb{Z}_p^{\lambda} \times$ (a grp of bdd exp).)

$\mathrm{Sel}_E(K_\infty) = \ker\left(H^1(K_\infty, E[p^\infty]) \to \bigoplus_v \mathcal{H}_v(K_\infty, E) \right)$ where v runs over primes of F and we do not define what the local conditions $\mathcal{H}_v(K_\infty, E)$ are.

Let Σ_0 be a finite set of primes of F ~~not~~ containing primes over p or ∞.

$\mathrm{Sel}_E^{\Sigma_0}(K_\infty) = \ker\left(H^1(K_\infty, E[p^\infty]) \longrightarrow \bigoplus_{v \notin \Sigma_0} \mathcal{H}_v(K_\infty, E) \right)$.

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$$\mathrm{Sel}_E^{\Sigma_0}(K_\infty) / \mathrm{Sel}_E(K_\infty) \simeq \bigoplus_{v \in \Sigma_0} \mathcal{H}_v(K_\infty, E).$$

Important Assumptions: ① $\Phi_{K/F} \subseteq \Sigma_0$

② $\mathrm{Sel}_E(K_\infty)[p]$ is finite. This mean that the group of bounded exp. above is a finite group. This is actually the assump. that the μ-inv. is 0.

Thm: Assume (a) $E(K)[p] = 0$

(b) $\tilde{E}_v(k_w)[p] = 0$ for $w \mid v \mid p$.

(c) $\Phi_{K/F} \subseteq \Sigma_0$

(d) $\mathrm{Sel}_E(K_\infty)[p]$ is finite.

Then

$X_E^{\Sigma_0}(K_\infty)$ is a projective $\mathbb{Z}_p[\Delta]$-module.

Thm: Assume only (c) and (d) above. Then $X_E^{\Sigma_0}(K_\infty)$ is a quasi-projective $\mathbb{Z}_p[\Delta]$-module.

Let X be a f.g. $\mathbb{Z}_p[\Delta]$-module. X is quasi-projective if X_1, X_2 exist, both projective, as $\mathbb{Z}_p[\Delta]$-modules s.t.

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X \longrightarrow 0$$

is exact when tensoring with $\mathbb{Q}_p$.

$\mathrm{Irr}_{\mathcal{F}}(\Delta)$ = irred. reps of Δ over $\mathcal{F}$ where $\mathcal{F}/\mathbb{Q}_p$ finite, $\mathcal{O} = \mathcal{O}_{\mathcal{F}}$, $f = \mathcal{O}/m$.

$\mathrm{Irr}_{\mathbb{F}}(\Delta)$ will also be considered.

Let ρ be a rep. $/\mathcal{F}$ of Δ. Then ρ can be realized over $\mathcal{O}$. Define $\tilde{\rho} = \rho$ modulo m. However, this depends on the realization over $\mathcal{O}$ chosen. The semi-simplification $\tilde{\rho}^{ss}$ is well-defined though.

Let $\sigma \in \mathrm{Irr}_{\mathcal{F}}(\Delta)$, define: $\lambda_X(\sigma) =$ mult. of σ in $X \otimes_{\mathbb{Z}_p} \mathcal{F}$.

<u>Basic Fact:</u> Suppose $\rho_1 \simeq \bigoplus_\sigma \sigma^{m_1(\sigma)}$, $\rho_2 = \bigoplus_\sigma \sigma^{m_2(\sigma)}$ be reps. of Δ over $\mathcal{F}$. Suppose X is projective $\mathbb{Z}_p[\Delta]$-module. Assume $\tilde{\rho}_1^{ss} \simeq \tilde{\rho}_2^{ss}$. Then

$$\sum_\sigma m_1(\sigma)\lambda_X(\sigma) = \sum_\sigma m_2(\sigma)\lambda_X(\sigma).$$

This is also true if X is only quasi-projective.
This can also be written as $\lambda_X(\rho_1) = \lambda_X(\rho_2)$.

Write if $X = X_E(K_\infty)$, $\lambda_X(\sigma) = \lambda_E(\sigma)$,
if $X = X_E^{\Sigma_0}(K_\infty)$, $\lambda_X(\sigma) = \lambda_E^{\Sigma_0}(\sigma)$.

<u>Note:</u> $\lambda_E^{\Sigma_0}(\sigma) - \lambda_E(\sigma) = \sum_{v \in \Sigma_0} \delta_v(\sigma)$ is something that can be studied.

$X = X_E(K) = \mathrm{Hom}(\mathrm{Sel}_E(K)_p, \mathbb{Q}_p/\mathbb{Z}_p)$. Write $\lambda_X(\sigma) = S_E(\sigma)$.

<u>Prop:</u> If $\sigma \in \mathrm{Irr}_{\mathcal{F}}(\Delta)$ is self-dual and orthogonal, then $S_E(\sigma) \equiv \lambda_E(\sigma) \pmod 2$. (Assuming Mazur's conjecture.)

We now discuss a motivating example. Take $F = \mathbb{Q}$. Let A = elliptic curve. Assume

$$Gal(\mathbb{Q}(A[p^{r+1}])/\mathbb{Q}) \simeq GL_2(\mathbb{Z}/p^{r+1}\mathbb{Z}).$$

$K_r \subseteq \mathbb{Q}(A[p^{r+1}])$ with $Gal(K_r/\mathbb{Q}) \simeq PGL_2(\mathbb{Z}/p^{r+1}\mathbb{Z})$.

Take $\Delta_r = Gal(K_r/\mathbb{Q})$, $\Delta_0 = Gal(K_0/\mathbb{Q}) \simeq PGL_2(\mathbb{Z}/p\mathbb{Z})$.

$\ker(\Delta_r \twoheadrightarrow \Delta_0) = p$-group.

Thus,

$$Irr_{\mathbb{F}_r}(\Delta_r) = Irr_{\mathbb{F}_r}(\Delta_0)$$

Let $\rho_1 = \sigma_{ST}^{(r)} \simeq Ind_{B_r}^{\Delta_r}(1_{B_r}) / Ind_{B_{r-1}}^{\Delta_{r-1}}(1_{B_{r-1}})$ $\quad B_r$ = Borel in $PGL_2(\mathbb{Z}/p\mathbb{Z})$

is the Steinberg representation at level r. We have

$$\tilde{\rho}_1^{ss} \simeq \tilde{\rho}_2^{ss} \simeq \bigoplus_{\sigma \in Irr_{\mathbb{F}}(\Delta)} \sigma^{n(\sigma)}.$$

where $\rho_2 = (\text{regular rep. of } \Delta_0)^{p^r}$

Then if X is a proj. $\mathbb{Z}_p[\Delta_r]$-module $\Rightarrow$

$$\begin{aligned} \lambda_X(\sigma_{ST}^{(r)}) &= p^r \lambda_X(\text{reg rep of } \Delta_0) \\ &= p^r \sum_\sigma n(\sigma)\lambda_X(\sigma). \\ &\equiv \lambda_X(\sigma_0) + \lambda_X(\sigma_1) + \lambda_X(\sigma_{p,1}) + \lambda_X(\sigma_{p,2}) \pmod 2. \end{aligned}$$

Take $X = X_E(K_{r,\infty})$, assume $\Phi_{K_r/\mathbb{Q}} = \phi$, $\Sigma_0 = \phi$, and $Sel_E(K_0)[p]$ is finite. Then the theorem imply X is quasi-proj. This implies

$$S_E(\sigma_{ST}^{(r)}) \equiv S_E(\sigma_0) + S_E(\sigma_1) + S_E(\sigma_{p,1}) + S_E(\sigma_{p,2}) \pmod 2.$$