Modular Forms and the Gross-Stark conjecture:

Joint work w/ H. Darmon and R. Pollack.

F = totally real field

H = CM abelian ext of F

$\wp$ = prime ideal of F that splits completely in H

$\chi : \mathrm{Gal}(H/F) \longrightarrow E^\times$, E fin. ext of $\mathbb{Q}_p$.

χ totally odd character, primitive.

$U_\wp = \{ h \in H^\times : |h|_w = 1 \ \forall w \nmid \wp \}$

$(U_\wp \otimes E)^{\chi^{-1}}$ is 1-dimensional as an E v.s. (u^χ a gen)

Fix a prime $\mathfrak{P}$ of H that is above $\wp$.

$U_\wp \subset H \subset H_\mathfrak{P}$

$H_\mathfrak{P}^\times \xrightarrow{\mathrm{ord}_\mathfrak{P}} \mathbb{Z}$ $\qquad$ $H_\mathfrak{P}^\times \xrightarrow{L_\mathfrak{P}} \mathbb{Z}_p$

$L_\mathfrak{P}(x) = \log_p \mathrm{Nm}_{H_\mathfrak{P}/\mathbb{Q}_p}(x)$

These extend to maps $\mathrm{ord}_\mathfrak{P}$ and $L_\mathfrak{P}$

$$U_\wp \otimes E \longrightarrow E$$

Def: $\mathcal{L}_{alg}(\chi) = \dfrac{L_\mathfrak{P}(u^\chi)}{\mathrm{ord}_\mathfrak{P}(u^\chi)}$

We can now view χ as a Ray class character with modulus divisible by q for all primes $q \mid p$, $q \neq \wp$.

$$L(\chi, s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_F} \frac{\chi(\mathfrak{a})}{(\mathrm{Nm}\,\mathfrak{a})^s} \qquad (\mathrm{Re}(s) > 1)$$

$L_p(\chi\omega, s)$ is a mero. fctn of $s \in \mathbb{Z}_p$, valued in E, with ω the Teichmuller character such that

$$L_p(\chi\omega, n) = \underbrace{L}_{\text{modulus div. by all primes dividing } \wp}(\chi\omega^n, n) \quad \forall n \in \mathbb{Z}_{\leq 0}.$$

$$\begin{aligned} L_p(\chi\omega, 0) &= L(\chi\omega^0, 0) \\ &= (1-\chi(\wp))\, L(\chi, 0). \\ &= 0 \cdot L(\chi, 0) \qquad \text{b/c } \wp \text{ splits completely.} \\ &= 0 \end{aligned}$$

Def: $\mathcal{L}_{an}(\chi) = \dfrac{L_p'(\chi\omega, 0)}{L(\chi, 0)}$

Note: We assume $L(\chi, 0) \neq 0$, i.e. no $q | p$, $q \neq \wp$, splits in H^χ.

Conj. (Gross): $\mathcal{L}_{alg}(\chi) = \mathcal{L}_{an}(\chi)$.

Gross proved this in the case $F = \mathbb{Q}$.

Notational simplification: $F = \mathbb{Q}$. Note everything said here will transfer to $F \neq \mathbb{Q}$ via Hilbert modular forms. So now $p = \wp$.

① Kummer Theory:

$$U^\chi \longleftrightarrow \gamma_\chi \in H_f^1(\mathbb{Q}, E(\chi)(1)) \quad \left(\begin{array}{l}\text{unram. away from} \\ p \text{ b/c coming from} \\ p\text{-unit}\end{array}\right)$$

$$\downarrow$$

$$\gamma_{\chi,p} \in H^1(\mathbb{Q}_p, E(1)) \quad \left(\begin{array}{l}\chi \text{ gone b/c } p \\ \text{splits completely}\end{array}\right)$$

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$\kappa \in H^1_f(\mathbb{Q}, E(\chi^{-1}))$ (unram. away from p, no condition at p)

$\rightsquigarrow$

$\kappa_p \in H^1(\mathbb{Q}_p, E)$

We have a cup product pairing.

Poitou-Tate: $\kappa_p \cup \gamma_{\chi,p} = 0 \in H^2(\mathbb{Q}_p, E(1)) \simeq E$

Unwinding all the identifications,

$$\kappa_p \cup \gamma_{\chi,p} = \kappa_p(\mathrm{Art}(u^\chi)) = 0. \qquad \mathrm{Art}: \mathbb{Q}_p^\times \to G_{\mathbb{Q}_p}^{ab}$$

$H^1(\mathbb{Q}_p, E) \simeq E^2$. So $H^1(\mathbb{Q}_p, E)$ has a natural basis κ^{ord}, κ^{log} defined by

$$\kappa^{ord}(\mathrm{Art}(u)) = \mathrm{ord}_p(u) \quad \text{and}$$

$$\kappa^{log}(\mathrm{Art}(u)) = \log_p(u).$$

Write $\kappa_p = x \cdot \kappa^{ord} + y \cdot \kappa^{log}$.

$\Rightarrow$

$$x \,\mathrm{ord}_p(u^\chi) + y \log_p(u^\chi) = 0.$$

Thus,

$$\mathcal{L}_{alg}(\chi) = \frac{-x}{y}.$$

So we have essentially removed the units from the conjecture.

So Gross' conjecture is equivalent to $\exists\ \kappa \in H^1_f(\mathbb{Q}, E(\chi^{-1}))$ s.t. $\kappa_p = -\mathcal{L}_{an}(\chi)\kappa^{ord} + \kappa^{log}$.

② Hida families

Working def: A Λ-adic form is a power series

$$F = \sum_{n=0}^{\infty} a_n q^n \qquad \text{with each} \quad a_n \in \Lambda = \{\text{Iwasawa fctns}\} \subset \left\{\begin{matrix}\text{p-adic anal. fctns} \\ \text{of var } k \in \mathbb{Z}_p\end{matrix}\right\}$$

s.t. $F(k) = \sum_{n=0}^{\infty} a_n(k) q^n$ is a classical modular form of wt k and character $\chi \omega^{1-k}$.

Let $\mathbb{T}$ = Hecke alg. of the space of Λ-adic forms.

Thm: Let $\mathcal{L}_{an}(\chi) + \mathcal{L}_{an}(\chi^{-1}) \neq 0$. Then $\exists$ Λ-alg. homom.

$$\mathbb{T} \longrightarrow E[\varepsilon]/\varepsilon^2$$

s.t.

$$T_\ell \longmapsto \Psi_1(\ell) + \Psi_2(\ell)$$

$$U_p \longmapsto 1 + \lambda \mathcal{L}_{an}(\chi)\varepsilon$$

where

$$\Psi_1, \Psi_2 : G_{\mathbb{Q}}^{ab} \longrightarrow \left(E[\varepsilon]/\varepsilon^2\right)^\times$$

$\Psi_1 \equiv 1$, $\Psi_2 \equiv \chi \pmod{\varepsilon}$ and

$$\lambda = \frac{\mathcal{L}_{an}(\chi)}{\mathcal{L}_{an}(\chi) + \mathcal{L}_{an}(\chi^{-1})},$$

$$\Psi_1|_{G_{\mathbb{Q}_p}} = 1 + (1-\lambda)\kappa' \varepsilon.$$

③

Thm: Given $\mathbb{T} \longrightarrow E[\varepsilon]/\varepsilon^2$ s.t.

$$T_\ell \longmapsto \Psi_1(\ell) + \Psi_2(\ell)$$

w/ $\Psi_1 = 1 + \Psi_1' \varepsilon \qquad \Psi_2 = \chi + \Psi_2' \varepsilon.$

$$U_p \longmapsto 1 + a_p' \varepsilon$$

then $\exists$ $\kappa \in H^1_f(\mathbb{Q}, E(\chi^{\pm 1}))$ s.t.

$$\kappa_p = -a_p' \kappa^{ord} + \kappa^{log} - \psi_1'|_{G_{\mathbb{Q}_p}}$$

<u>Pf</u>: Follows Ribet's method and use Galois reps associated to Λ-adic forms. ■

Combining theorems one obtains a class κ s.t.

$$\kappa_p = \lambda(-\mathcal{L}_{an}(\chi)\kappa^{ord} + \kappa^{log}).$$

Assuming $\lambda \neq 0$ we are done.

<u>Thm:</u> If Leopoldt's conjecture holds for F and $\mathcal{L}_{an}(\chi) + \mathcal{L}_{an}(\chi^{-1}) \neq 0$, then Gross' conjecture is true for χ.

<u>Cor:</u> If F is real quadratic and H is a narrow ring class field ext of F, then Gross' conjecture is true ~~for~~ H/F.

Idea for proof of Thm constructing Λ-alg. homom:

$$E_k^*(1, \chi\omega^{1-k}) = \frac{L_p(\chi\omega, 1-k)}{2} + \sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ p\nmid d}} \chi(d)\langle d\rangle^{k-1}\Big) q^n$$

$$\langle d\rangle = \frac{d}{\omega(d)}.$$

$$C(G)E_k^* - \frac{L_p(\chi\omega, 1-k)}{2}\underbrace{G_k^*}_{\text{prod of 2 E.S.}} \quad \text{as in Ribet.}$$