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Tate-Shafarevich Groups of elliptic curves over anti-cyclo $\mathbb{Z}_p$-extns:

E elliptic curve $/\mathbb{Q}$

It is known if $\quad \mathrm{rk}_{an}\, E/\mathbb{Q} \leq 1 \Rightarrow Ш(E/\mathbb{Q})$ is finite.

$Ш(E/\mathbb{Q})$ finite $\Leftrightarrow Ш(E/\mathbb{Q})[p^\infty]$ finite $\forall$ prime p.

Fix a prime p and $F/\mathbb{Q}$ a Galois ext.
Then

$$Ш(E/F)[p^\infty] \text{ is a } \mathbb{Z}_p[\mathrm{Gal}(F/\mathbb{Q})]\text{-module.}$$

One then tries to choose F appropriately.
$F = \mathbb{Q}_\infty$ - cyclo $\mathbb{Z}_p$-ext. Then $Ш(E/\mathbb{Q}_\infty)[p^\infty]$ is a $\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]$-module. $\Lambda := \mathbb{Z}_p[[T]]$. It is known that

$$Ш(E/\mathbb{Q}_\infty)[p^\infty]^\wedge := \mathrm{Hom}(Ш(E/\mathbb{Q}_\infty)[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p)$$

is f.g over Λ.
So we have an exact sequence

$$\text{fin. grp} \to Ш(E/\mathbb{Q}_\infty)[p^\infty]^\wedge \to \Lambda^r \oplus \Lambda/f_1 \oplus \cdots \oplus \Lambda/f_s \to \text{fin. grp.}$$

r is uniquely determined,
$r = \mathrm{cork}_\Lambda\, Ш(E/\mathbb{Q}_\infty)[p^\infty]$.

If p is ord, then $r = 0$. (Kato '04, Rubin '88)
If p is s.s, then $r \geq 1$ (Schneider '84, Rohrlich '85, Kato '04)

Kurihara '02 $\quad \mathrm{rk}_{an}\, E/\mathbb{Q} = 0 \quad \Rightarrow Ш(E/\mathbb{Q}_\infty)[p^\infty]^\wedge \simeq \Lambda$.
Pollack '05 $\quad \mathrm{rk}_{alg}\, E/\mathbb{Q} = 0$

Since $\mathbb{Q}_\infty$ is only $\mathbb{Z}_p$-ext of $\mathbb{Q}$, we are done in that case. Now let $K/\mathbb{Q}$ be imag. quad ext. s.t. every prime $\ell \mid \mathrm{cond}\, E/\mathbb{Q}$ splits. Let K_∞/K be the anticyclo $\mathbb{Z}_p$-ext, i.e.,

$$\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p$$

$$\mathrm{Gal}(K_\infty/\mathbb{Q}) = \text{prodihedral (c.c act on every elt by inverting it)}$$

$$\text{Ш}(E/K_\infty)[p^\infty] \text{ is a } \mathbb{Z}_p[\![\underbrace{\mathrm{Gal}(K_\infty/K)}_{\Gamma}]\!]\text{-module.}$$

What is $\mathrm{cork}_\Lambda\, \text{Ш}(E/K_\infty)[p^\infty]$?

$\left.\begin{array}{l} p \text{ ord:} \\ p \text{ s.s:} \end{array}\right\} \mathrm{cork}_\Lambda\, \text{Ш}(E/K_\infty)[p^\infty] = 0$ Bertolini '95, Ciperiani '09

<u>Sketch of proof in s.s case:</u>

$$0 \to E(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1_{Sel}(K_\infty, E[p^\infty]) \to \text{Ш}(E/K_\infty)[p^\infty] \to 0.$$

If $\mathrm{cork}_\Lambda\, E(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p = \mathrm{cork}_\Lambda\, H^1_{Sel}(K_\infty, E[p^\infty])$, then $\mathrm{cork}_\Lambda\, \text{Ш}(E/K_\infty) = 0$. One shows they both have cork 2 which gives the result.

$$0 \to E(K_n)/p^n \to H^1_{Sel}(K_n, E[p^n]) \to \text{Ш}(E/K_n)[p^n] \to 0$$

$$\text{Ш}(E/K_\infty)[p^\infty] = \varinjlim \text{Ш}(E/K_n)[p^n].$$

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$$H^1_{Sel}(K_n, E[p^n]) := \operatorname{Ker}\left(H^1(K_n, E[p^n]) \to \prod_{\lambda} H^1(K_{n,\lambda}, E) \right).$$

Need to enlarge $H^1_{Sel}(K_n, E[p^n])$: choose Q_n a set of rational primes s.t.

1) $\ell \in Q_n$ inert in $K/\mathbb{Q}$ and $p \notin Q_n$
2) $E(K_\ell)[p^\infty] = E(\bar{K}_\ell)[p^n]$ $\quad K_\ell$ completion of K at ℓ
3) $H^1_{Sel}(K, E[p]) \hookrightarrow \prod_{\ell \in Q_n} H^1(K_\ell, E[p])$
4) $\# Q_n < \infty$ and indep. of n.

Set $t = \# Q_n$.

Rmk: Bertolini: $\prod_{\lambda | \ell} H^1(K_{n,\lambda}, E[p^n]) \simeq R_n^2$ where

$$R_n \simeq (\mathbb{Z}/p^n\mathbb{Z})[\operatorname{Gal}(K_n/K)]$$

$\forall\ \ell \in Q_n$, $\forall m \geq n$.

$$H^1_{Sel_{p \cup Q_m}}(K_n, E[p^n]) = \operatorname{Ker}\left(H^1(K_n, E[p]) \to \prod_{\substack{\lambda \nmid \ell \in Q_m \\ \lambda \nmid p}} H^1(K_{n,\lambda}, E) \right)$$

this is a R_n-module.

We have

$$H^1_{Sel_{p \cup Q_m}}(K_n, E[p^n]) \qquad \forall m \geq n.$$

One can show

$$\# H^1_{Sel_{p \cup Q_m}}(K_n, E[p^n]) = \# R_n^{2t+2}.$$

Thus, infinitely many $H^1_{Sel_{p \cup Q_m}}(K_n, E[p^n])$ are ism.

One has $E(K^{\text{solvable}})[p]$ is trivial $\forall m \geq n$ $\quad (p > 3)$

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Thus, we have an injection

$$H^1_{Sel_{p\cup Q_m}}(K_n, E[p^n]) \hookrightarrow H^1_{Sel_{p\cup Q_m}}(K_{n+1}, E[p^{n+1}]).$$

So we can choose a sequence $\{k_n\}$ s.t.

$$\begin{tikzcd} H^1_{Sel_{p\cup Q_{k_n}}}(K_n, E[p^n]) \arrow[r, phantom, "\simeq"] \arrow[dr, hook] & H^1_{Sel_{p\cup Q_{k_{n+1}}}}(K_n, E[p^n]) \arrow[d, hook] \\ & H^1_{Sel_{p\cup Q_{k_{n+1}}}}(K_{n+1}, E[p^{n+1}]). \end{tikzcd}$$

Thus, we can consider the direct limit

$$\mathcal{M} := \varinjlim H^1_{Sel_{p\cup Q_{k_n}}}(K_n, E[p^n]).$$

Thm (C.-Wiles): $\hat{\mathcal{M}} \simeq \Lambda^{2t+2}$.

$$\left\{\begin{array}{l} \mathcal{M}[g^{p^n}-1, p^n] \hookrightarrow H^1_{Sel_{p\cup Q_{k_m}}}(K_n, E[p^n]) \\ \# H^1_{Sel_{p\cup Q_{k_m}}}(K_n, E[p^n]) = \# R_n. \end{array}\right.$$

$$\Rightarrow \quad H^1_{Sel_{p\cup Q_{k_m}}}(K_n, E[p^n]) \simeq R_n^{2t+2} \qquad \forall m \geq n.$$

$$\begin{array}{ccc} H^1_{Sel_{p\cup Q_{k_m}}}(K_n, E[p^n]) & \xrightarrow{\psi_n^m} & \prod_{\lambda | \ell \in Q_{k_m}} H^1(K_{n,\lambda}, E)[p^n] \\ \wr\vert & & \wr\vert \\ \varphi_n^m : R_n^{2t+2} & & R_n^{2t} \end{array} \qquad \forall m \geq n.$$

$$\operatorname{Ker} \psi_n^m = H^1_{Sel_p}(K_n, E[p^n])$$

p s.s. so $\varinjlim H^1_{Sel_p}(K_n, E[p^n]) = H^1_{Sel}(K_\infty, E[p^\infty])$.

Restrict to a subsequence so that

$$\varphi_n^m = \varphi_n^n =: \varphi_n.$$

$$\begin{tikzcd} H^1_{Sel_{p\cup Q_{k_m}}}(K_n, E[p^n]) \arrow[r] \arrow[d, hook] & \prod_{\lambda \mid \ell \in Q_{k_m}} H^1(K_{n,\lambda}, E)[p^n] \arrow[d, hook] \\ H^1_{Sel_{p\cup Q_{k_m}}}(K_{n+1}, E[p^{n+1}]) \arrow[r] & \prod_{\lambda \mid \ell \in Q_{k_m}} H^1(K_{n+1,\lambda}, E)[p^{n+1}] \end{tikzcd}$$

$m > 0$

$\Rightarrow$

$$\begin{tikzcd} R_n^{2t+2} \arrow[r, "\varphi_n"] \arrow[d, hook] & R_n^{2t} \arrow[d, hook] \\ R_{n+1}^{2t+2} \arrow[r, "\varphi_{n+1}"] & R_{n+1}^{2t} \end{tikzcd}$$

Define $\varphi = \varinjlim \varphi_n : \varinjlim R_n^{2t+2} \longrightarrow \varinjlim R_n^{2t}$

$$\hat{\Lambda}^{2t+2} \longrightarrow \hat{\Lambda}^{2t}$$

Observe $\operatorname{Ker} \varphi_n \simeq H^1_{Sel_p}(K_n, E[p^n])$

$\Rightarrow \operatorname{Ker} \varphi \simeq \varinjlim H^1_{Sel_p}(K_n, E[p^n])$

~~not nec. usual maps~~

trans. maps are inj.

$\Rightarrow \operatorname{cork}_\Lambda \operatorname{Ker}\varphi = \operatorname{cork}_\Lambda H^1_{Sel}(K_\infty, E[p^\infty])$.

For each $\ell \in Q_m$ $\overset{\text{construct}}{\rightsquigarrow}$ Kolyvagin classes

$\mathrm{Kol}_{Q_m}(K_n, E[p^n]) \in H^1_{Sel_{p_0 Q_m}}(K_n, E[p^n])$

These give that $\operatorname{cork}_\Lambda \operatorname{Im}\varphi = 2t$.

$\operatorname{cork}_\Lambda E(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 2$ Cornut-Vatsal : Heegner pt

$\alpha_n \in E(K_n) \setminus E(K_n)_{tors}$

$\forall n \gg 0$.

$\Rightarrow \operatorname{cork}_\Lambda Ш(E/K_\infty)[p^\infty] = 0$.