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Restricted Selmer groups and special values of p-adic L-functions:

$E/\mathbb{Q}$ elliptic curve with CM by $\mathcal{O}_K \subset K$

r = rank of E

$p \neq 2$ prime of good ordinary prime; thus p splits in $\mathcal{O}_K$, $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$, $(\varpi) = \mathfrak{p}$, $\varpi^* = \mathfrak{p}^*$.

$G_\infty = Gal(K(E[p^\infty])/K)$

$\quad \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \Delta \qquad \psi$

$\psi : G_\infty \longrightarrow Aut(E[\varpi^\infty]) \simeq \mathcal{O}_{K_\mathfrak{p}}^\times \simeq \mathbb{Z}_p^\times$

$\psi^* : G_\infty \rightarrow Aut(E[\varpi^{*\infty}]) \simeq \mathcal{O}_{K,\mathfrak{p}^*}^\times \simeq \mathbb{Z}_p^\times$.

Katz 2-variable p-adic L-function:

$\mathcal{L}_\mathfrak{p} : Hom(G_\infty, \mathbb{Z}_p^\times) \longrightarrow \mathbb{C}_p$: Defined via p-adic interpolation:

$$\mathcal{L}_\mathfrak{p}(\psi^k \psi^{*j}) \doteq L(\psi^{k}\bar{\psi}^{-j}, 0) \qquad k \geq 0,\ -k < j \leq 0$$

where $\doteq$ means equality up to an explicit non-zero constant.

"classical pt" ψ

Define $L_\mathfrak{p}(s) := \mathcal{L}_\mathfrak{p}(\psi\langle\psi\rangle^{s-1})$ where $\langle\psi\rangle : G_\infty \xrightarrow{\psi} \mathbb{Z}_p^\times \rightarrow 1+p\mathbb{Z}_p$

$s \in \mathbb{Z}_p$

Conjecture: ($\mathfrak{p}$-adic BSD):

- $\operatorname{ord}_{s=1} L_\mathfrak{p}(s) = r$
- $\mathcal{L}_\mathfrak{p}^{(r)}(\psi) \doteq |Ш| \cdot R_\mathfrak{p}$

$\langle , \rangle_\mathfrak{p} : Sel(K,T) \times Sel(K,T^*) \longrightarrow \mathbb{Q}_p$ $\quad$ $\mathfrak{p}$-adic height pairing of Perrin-Riou.

$R_\mathfrak{p} = \det \langle x_i, x_j\rangle_\mathfrak{p}$ where $\{x_i\}$ basis of $E(K)/tors$.

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Rubin: True if $\operatorname{ord}_{s=1} L(E,s) = 0$ or 1.

"non-classical pts": What is the behavior of $\mathcal{L}_\wp$ at ψ^*?

$$L^*_\wp(s) := \mathcal{L}_\wp(\psi^* \langle \psi^* \rangle^{s-1}).$$

Conjecture (Rubin): Suppose $r \geq 1$.

- $\operatorname{ord}_{s=1} L^*_\wp(s) = r-1$.
- $\mathcal{L}_\wp^{(r-1)}(\psi^*) \doteq |Ш| \, R^*_\wp$

where

$$R^*_\wp = \det(\langle x_i, x_j \rangle_\wp + \log_{E_\wp}(x_i) \log_{E_\wp}(x_j)) - \det(\langle x_i, x_j \rangle_\wp).$$

$r=1$: Conj. is true

$$\mathcal{L}_\wp(\psi^*) \doteq \log_{E_\wp}(y)^2 \qquad \langle y \rangle = E(\mathbb{Q})/_{\text{tors}}$$

$\rightsquigarrow$ p-adic construction of a global rational pt.

Questions:

- What happens when $r=0$?

  The functional equation for $\mathcal{L}_\wp$ predicts $\mathcal{L}_\wp(\psi^*) = \emptyset$.
- Why does $R^*_\wp$ arise?

Idea: Find a Selmer group that controls the behavior of $L^*_\wp(s)$.

F/K any finite extension. Define the restricted Selmer group

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$$\Sigma_{\wp^*}(F, E[\varpi^* \infty]) = \{ c \in H^1(F, E[\varpi^*\infty]) : \mathrm{loc}_v(c) = 0 \quad \forall v \nmid \wp \}.$$

There are compact versions: $\Sigma_{\wp^*}(F, T^*)$, $\Sigma_{\wp}(F, T)$.

Properties: Suppose Ш$(K)[p]$ is finite

- $\mathrm{rk}_{\mathcal{O}_{K,\wp^*}} (\Sigma_{\wp^*}(K, T^*)) = |r-1|$

- $\exists$ a $\wp$-adic height pairing

  $[\ ,\ ]_{\wp^*} : \Sigma_{\wp}(K, T) \times \Sigma_{\wp^*}(K, T^*) \longrightarrow \mathbb{Z}_p$.

  $\rightsquigarrow R_{\wp^*}$ p-adic regulator

- If $r \geq 1$, then $\Sigma_{\wp^*}(K, T^*) = \mathrm{Ker}(\mathrm{Sel}(K, T^*) \to E(K_{\wp^*}) \otimes \mathcal{O}_{K,\wp^*})$.

Thm: Suppose Ш$(K)[p]$ is finite and $R_{\wp^*} \neq 0$. Then

- $\mathrm{ord}_{s=1} L^*_{\wp}(s) = |r-1|$
- $\mathcal{L}_{\wp}^{(|r-1|)}(\psi^*) \sim |\Sigma_{\wp^*}(K, E[\varpi^* \infty]) / \text{max div gp}| \cdot R_{\wp^*}$.

  ($\sim$: up to p-adic unit)

One uses main conjectures for 2 variable p-adic L-functions to prove this which is where the ambiguity up to p-adic unit arises.

Now suppose $r = 0$. Then p-adic BSD is known to hold.

Thm: Suppose $\mathcal{L}_{\wp}(\psi) \neq 0$.

(a) The p-adic height pairing $[\ ,\ ]$ is non-degenerate and $\mathcal{L}_{\wp}'(\psi^*) \neq 0$.

(b) Suppose that $y \in \Sigma_{\mathfrak{p}}(K,T)$, $y^* \in \Sigma_{\mathfrak{p}^*}(K,T^*)$ are of infinite order. Then

$$\frac{\Omega_{\mathfrak{p}} \cdot \mathcal{L}_{\mathfrak{p}}'(\psi^*)}{[y,y^*]_{K,\mathfrak{p}^*}} \doteq \frac{\mathcal{L}_{\mathfrak{p}^*}(\psi^*)}{\Omega_{\mathfrak{p}^*}\, \exp_{\mathfrak{p}}^*(y^*)\, \exp_{\mathfrak{p}^*}^*(y)}$$

where

$$\exp_{\mathfrak{p}}^* : \Sigma_{\mathfrak{p}^*}(K,T^*) \longrightarrow \mathbb{Q}_p$$

are the Bloch-Kato dual exponential maps.

Ideas of proof:

- Elliptic units $\rightsquigarrow$ canonical elements

$$s_{\mathfrak{p}} \in \Sigma_{\mathfrak{p}}(K,T), \quad s_{\mathfrak{p}^*} \in \Sigma_{\mathfrak{p}^*}(K,T^*)$$

$s_{\mathfrak{p}^*}$ is of infinite order $\Leftrightarrow$ $\mathcal{L}_{\mathfrak{p}}'(\psi^*) \neq 0$ and $R_{\mathfrak{p}}^* \neq 0$.

- $\exp_{\mathfrak{p}}^*(s_{\mathfrak{p}^*}) \doteq \mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$.

  $\Rightarrow s_{\mathfrak{p}^*}$ is of infinite order.

  $[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*} \doteq \mathcal{L}_{\mathfrak{p}}(\psi) \cdot \mathcal{L}_{\mathfrak{p}}'(\psi^*)$

- $$\frac{\exp_{\mathfrak{p}}^*(s_{\mathfrak{p}^*})\, \exp_{\mathfrak{p}^*}^*(s_{\mathfrak{p}})}{[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*}} = \frac{\exp_{\mathfrak{p}}^*(y^*) \cdot \exp_{\mathfrak{p}^*}^*(y)}{[y,y^*]_{K,\mathfrak{p}^*}}.$$

The theorem follows from this and p-adic BSD.

Remarks on higher weight:

Let $k \geq 0$ be an integer, and set $\phi_k := \psi^{k+1}(\psi^*)^{-k}$ and $\phi_k^* := \psi^k(\psi^*)^{1-k}$.

ϕ_k lies in the range of interpolation of $\mathcal{L}_\wp$.

$\mathcal{L}_\wp(\phi_k) \doteq L(\bar{\psi}^{2k+1}, k+1)$

$k=0$: BSD

$k \geq 1$: Bloch-Kato, Perrin-Riou, Beilinson.

- ϕ_k^* lies outside the range of interpolation of $\mathcal{L}_\wp$.

  $k \geq 1$: have not previously been studied.

If $W(\psi) = +1$ (root number): Then $\mathcal{L}_\wp(\phi_k) \neq 0$ for all but finitely many k (Greenberg, Rohrlich).

Have

$$\mathcal{L}_\wp(\phi_k) \doteq \mathcal{L}_\wp'(\phi_k^*).$$

If $W(\psi) = -1$: Then $\mathcal{L}_\wp(\phi_k) = 0$ for all k, and $\mathcal{L}_\wp'(\phi_k) \neq 0$ for all but finitely many k.

Have

$$\mathcal{L}_\wp'(\phi_k) \doteq \mathcal{L}_\wp(\phi_k^*).$$

Forthcoming work Bertolini-Darmon-Prasanna: does the $W(\psi) = -1$ case in terms of geometry.