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Mean Values of multiplicative functions and applications:

A multiplicative function is one that satisfies $f(mn) = f(m)f(n)$ $(m,n)=1$

<u>Example:</u> Up to x;
$$\begin{cases} f(1) = 1 & \\ f(p) = 0 & p \le x/2 \\ f(p) = ? & x/2 \le p \le x \end{cases}$$

The most you hope to save is $\frac{1}{\log x}$ over the trivial bound.

We would like to understand

$$\frac{1}{x}\sum_{n\le x} f(n).$$

Write

$$f(n) = \sum_{d|n} g(d) \qquad g(p) = f(p) - 1$$

As then

$$\frac{1}{x}\sum_{n\le x} f(n) = \frac{1}{x}\sum_{d\le x} g(d)\left[\frac{x}{d}\right]$$

$$\approx \sum_{d\le x}\frac{g(d)}{d}$$

$$\approx \prod_p \left(1 + \frac{g(p)}{p} + \dots\right)$$

$$\approx \exp\Big(\sum_{p\le x}\frac{f(p)-1}{p}\Big).$$

<u>Heuristic:</u> $\frac{1}{x}\sum f(n)$ compare with $\frac{1}{\log x}\exp\Big(\sum_{p\le x}\frac{f(p)}{p}\Big)$

$$\frac{1}{\log x}\prod\left(1 + \frac{f(p)}{p} + \dots\right)$$

Example: ① Smooth numbers

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$$f(n) = \begin{cases} 1 & \text{if} \quad p \mid n \Rightarrow p \le x^{\frac{1}{u}} \\ 0 & \text{o/w} \end{cases}$$

$\frac{1}{x} \sum_{n \le x} f(n) \sim \rho(u)$, but Euler product is of size $\frac{1}{u}$.

② $f(n) = n^{i\alpha}$ $\quad \alpha \in \mathbb{R} \setminus \{0\}$

$\sum_{n \le x} n^{i\alpha} \sim \frac{x^{1+i\alpha}}{1+i\alpha}$, but $\sum_{p \le x} \frac{p^{i\alpha}}{p} = O(1)$.

Erdős - Wintner: If f is mult. with $-1 \le f(n) \le 1$, does

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) \quad \text{exist?}$$

Easy case: $\sum_p \frac{1 - f(p)}{p}$ converges, the answer is yes.

Hard case: $\sum \frac{1 - f(p)}{p}$ diverges $\Rightarrow$ mean value $\to 0$ (Wirsing)

($\approx$ Prime Number Theorem)

Idea: $\sum_{n \le x} f(n) \log n \approx \log x \sum_{n \le x} f(n)$

$$\approx \sum_{p \le x} (\log p) f(p) \sum_{m \le x/p} f(m)$$

$$\log n = \sum_{p \mid n} \log p$$

Example: $\left| \sum_{n \le x} f(n) \right| \le \frac{1}{\log x} \int_2^x \left| \sum_{m \le x/t} f(m) \right| dt$

For complex valued functions:

$$\left| \frac{1}{x} \sum_{n \le x} f(n) \right|$$ is large only if for some $\alpha \in \mathbb{R}$, $f(p) \approx p^{i\alpha}$ for most primes p.

$$\left| \frac{1}{x} \sum_{n \le x} f(n) \right| \ll \exp\left(- \min_{|\alpha| \le (\log x)^{100}} \sum_{p \le x} \frac{1 - \mathrm{Re}\, f(p) p^{-i\alpha}}{p} \right)$$

$$\mathbb{D}(f,g;x)^2 = \sum_{p \le x} \frac{1 - \mathrm{Re}\, f(p)\overline{g(p)}}{p}$$. This defines a metric:

$$\mathbb{D}(f_1, g_1; x) + \mathbb{D}(f_2, g_2; x) \geq \mathbb{D}(f_1 f_2, g_1 g_2; x).$$

$\mathbb{D}(1, p^{it}; x)$ is large by the PNT.

$\mathbb{D}(\mu, p^{it}; x)$ is large by the PNT.

$\chi \bmod q$, $\mathbb{D}(1, \chi; q)$ We don't know how to say this is large.

<u>Linnik's Thm</u>: For characters up to conductor Q, with

$$\mathbb{D}(1, \chi; q) \le A.$$

There are at most $C(A)$ such characters.

$\uparrow$
constant depending on A

$\mathbb{D}(\mu, \chi; q)$? This is the Siegel zero problem.

① ψ = cubic character fixed, say conductor 7

χ_d = quadratic character of cond. d, d large.

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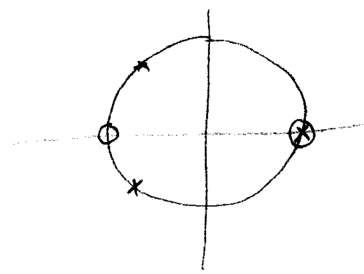
$$\sum_{n \le x} \psi \chi_d(n) \ll \frac{x}{(\log x)^{\theta}}, \quad \theta > 0.$$

Sketch: Need a lower bound for $\mathbb{D}(\psi\chi_d(p), p^{i\alpha}; x)$ for all $|\alpha| \le (\log x)^{100}$. Using triangle inequality

$$\ge \frac{1}{6}\mathbb{D}(1, p^{6i\alpha}; x) \quad \text{is large unless } \alpha \approx 0.$$

$\alpha = 0$:

$\mathbb{D}(\psi\chi_d, 1; x)$ is large



$\times = \psi$

$\circ = \chi$

quadratic character can't pretend to be cubic ...

② $L(s, \pi)$ $\quad$ π auto form on $GL(n)$ that is self-dual.

$$= \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} \qquad a_\pi(n) = \text{real}.$$

$L(1+it, \pi)$ $\quad$ π fixed, t = large.

Once can show

$$L(1+it, \pi) \ll (\log|t|)^{\theta} \quad \text{for some } \theta < 1.$$

This bound is inherited from Vinogradov.

③ Pólya-Vinogradov inequality (Granville-S)

$$\sum_{n \le q\alpha} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{\substack{m \in \mathbb{Z} \\ m \ne 0}} \frac{\overline{\chi}(m)}{m}(1 - e(-m\alpha))$$

$$\ll (\sqrt{q}\log q) \quad (\text{on GRH} \ll \sqrt{q}\log\log q)$$

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If χ has odd order g, then

$$\Big|\sum_{n \le q\alpha} \chi(n)\Big| \ll \sqrt{q}\,(\log q)^{1-\frac{\delta_g}{2}+o(1)}$$

$$\longrightarrow (\log q)^{1-\delta_g+o(1)} \text{ by Goldmakher.}$$

$$\delta_g = 1 - \frac{\sin(\pi/g)}{\pi/g}$$

$$\sum_m \bar{\chi}(m)\, e(m\alpha)$$

α is on minor arc — Dabussi, Delange, Montgomery - Vaughn

α is on a major arc, $\alpha \approx b/r$, r small

$$e\left(\tfrac{mb}{r}\right) \longrightarrow \sum_{\psi \,(\mathrm{mod}\, r)} (\cdots)$$

Get large contributions only if $\mathbb{D}(\chi,\psi;-)$ is small. If ψ exists, it is unique $\mathbb{D}(\chi,\psi_1)$ & $\mathbb{D}(\chi,\psi_2)$ are small $\Rightarrow$ $\mathbb{D}(\psi_1,\psi_2)$ is small.

④ <u>Weak Subconvexity</u>: $L(s,\pi) = \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s}$ π auto form on $GL(k)$.

Understand $L(\tfrac12,\pi)$ bound?

Convexity bound gives $\ll C(\pi)^{1/4}$. $C(\pi)$ = conductor of π.

Weak subconvexity $\ll \dfrac{C(\pi)^{1/4}}{(\log C(\pi))^{1-\varepsilon}}$