Some background on elliptic curves and Galois cohomology

Jim Brown

September 19, 2009

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3. A nonsingular plane projective curve E of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

More familiarly, if $\mathrm{char}\, k \neq 2, 3$, one can write an elliptic curve in the form

$$Y^2 = X^3 + aX + b$$

where $\Delta = 4a^3 + 27b^2 \neq 0$ (along with a point at infinity $(0 : 1 : 0)$.)

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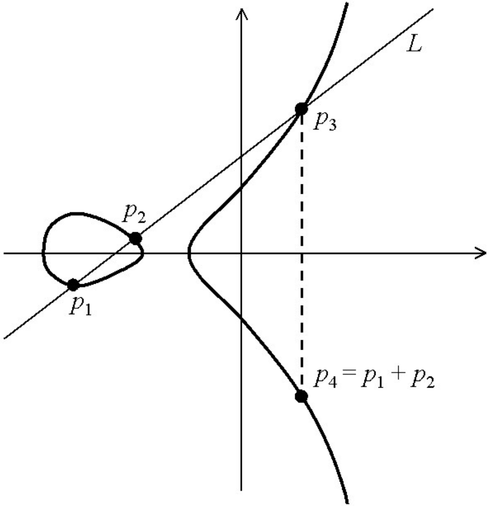

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2. For general number fields this is contained the thesis of Weil (1928). (He actually proved: given any nonsingular projective curve C over a number field k, one has $\mathrm{Pic}^0(C)$ is finitely generated.)

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To prove the Weak Mordell-Weil theorem one uses Galois cohomology, which we now review.

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Note that one has a natural action of G_k on $E(k^{\mathrm{al}})$ and on $E(k^{\mathrm{al}})[n]$ given by $\sigma \cdot (x, y) = (x^\sigma, y^\sigma)$.

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For $n = 0$, set

$$H^0(G, M) = M^G = \{m \in M : m^\sigma = m \text{ for all } \sigma \in G\}.$$

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A *principal crossed homomorphism* is a continuous homomorphism $f : G \to M$ satisfying

$$f(\sigma) = m^{\sigma} - m$$

for some fixed $m \in M$ and all $\sigma \in G$.

Define

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Theorem

Given an exact sequence of G-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

there is a canonical exact sequence

$$0 \to M_1^G \to M_2^G \to M_3^G \to H^1(G, M_1) \to H^1(G, M_2) \to H^1(G, M_3).$$

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In general we write $H^n(k, M)$ to denote $H^n(G_k, M)$.

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Which in turn gives an exact sequence:

$$0 \to E(k)[n] \to E(k) \xrightarrow{n} E(k) \to H^1(k, E(k^{\mathrm{al}})[n]) \\ \to H^1(k, E(k^{\mathrm{al}})) \xrightarrow{n} H^1(k, E(k^{\mathrm{al}})).$$

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Goal: Replace $H^1(k, E(k^{\mathrm{al}})[n])$ with a group we can show is finite and contains the image of $E(k)/nE(k)$.

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For each place v one obtains a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(k)/nE(k) & \longrightarrow & H^1(k, E(k^{\mathrm{al}})[n]) & \longrightarrow & H^1(k, E(k^{\mathrm{al}}))[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E(k_v)/nE(k_v) & \longrightarrow & H^1(k_v, E(k_v^{\mathrm{al}})[n]) & \longrightarrow & H^1(k_v, E(k_v^{\mathrm{al}}))[n] & \longrightarrow & 0. \end{array}$$

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Define the n-Selmer group by

$$\begin{aligned} \mathrm{Sel}_n(E/k) &= \{c \in H^1(k, E(k^{\mathrm{al}})[n]) : \ \forall v,\ c_v \text{ comes from } E(k_v)\} \\ &= \ker\left(H^1(k, E(k^{\mathrm{al}})[n]) \to \prod_v H^1(k_v, E(k_v^{\mathrm{al}})) \right). \end{aligned}$$

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1. It is conjectured that Ш is finite, but it is not known in general.
2. There is a precise (conjectural) relationship between the order of Ш and the rank of $E(k)$.
3. With the proper geometric interpretation Ш provides a measure of the failure of the local-global principle.

Using the exact sequence

$$0 \to E(k)/nE(k) \to H^1(k, E(k^{\mathrm{al}})[n]) \to H^1(k, E(k^{\mathrm{al}}))[n] \to 0$$

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and the kernel-cokernel exact sequence we obtain:

$$0 \to E(k)/nE(k) \to \mathrm{Sel}_n(E/k) \to Ш(E/k)[n] \to 0.$$

Finally, in the case of elliptic curves one can show $\mathrm{Sel}_n(E/k)$ is finite. In fact, it is actually computable! Thus we obtain the weak Mordell-Weil theorem.

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If $\widetilde{E}_v$ is a nodal cubic and the tangent lines at the node are not defined over $\mathbb{F}_v$, we say E has non-split multiplicative reduction at v.

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For $v \notin S$, set $a_v = 1 + q_v - \#\widetilde{E}_v(\mathbb{F}_v)$.

For such v, set $L_v(E/k, s) = (1 - a_v q_v^{-s} + q_v^{1-2s})^{-1}$.

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$$L_v(E/k, s) = \begin{cases} 1 & E \text{ has additive reduction} \\ (1 - q_v^{-s})^{-1} & E \text{ has split multiplicative reduction} \\ (1 + q_v^{-s})^{-1} & E \text{ has non-split multiplicative reduction} \end{cases}$$

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Once completed, the L-function has the usual properties one would expect.

Note that one can actually write $L_v(E/k, s)$ for any finite v as:

$$L_v(E/k, s) = \det(1 - \sigma_v^{-1} q_v^{-s} \mid (T_\ell(E)^\vee)^{I_v})^{-1}.$$

Weak BSD Conjecture

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The strong form of the conjecture that gives the first coefficient in the Taylor expansion of $L(E/\mathbb{Q}, s)$ around $s = 1$ will be discussed in the following talk. Note that it contains the order of the Shafarevich-Tate group!