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PS1

On the degree 5 L-function for GSp(4):

Hecke-Weil Converse Theorem:

$$\mathfrak{H}_1 = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$$

modular form

$$f : \mathfrak{H} \longrightarrow \mathbb{C} \quad \text{holomorphic}$$

$$\Gamma = SL_2(\mathbb{Z}) \qquad \gamma \in \Gamma, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

$$f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}$$ cusp form gives a sequence

$$\{a(n)\}_{n \geq 1}.$$

We would like to go the other direction, given a sequence $\{a(n)\}_{n\geq 1}$, when is $\sum a(n) e^{2\pi i n z}$ a modular form?

$$L(s,f) = \sum_{n \geq 1} \frac{a(n)}{n^s} = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}.$$

$$\Lambda(s,f) = \Gamma(s)(2\pi)^{-s} L(s,f)$$

Theorem (Hecke): Suppose we have a sequence $\{a(n)\}_{n=1}^{\infty}$

and

- $|a(n)| = O(n^{\kappa})$
- $\Lambda(s,f)$ analytic
- bounded in vertical strips

- $\Lambda(s,f)$ has a functional equation $s \leftrightarrow k-s$.

Then

$$f(z) = \sum_{n \geq 1} a(n) q^n$$

is a modular form.

<u>Imai's Converse Theorem & Siegel Modular Forms:</u>

$$J = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right)$$

$$Sp_4(\mathbb{R}) = \{ g \in SL_4 \mid {}^t g J g = J \}$$

$$\mathfrak{h}_2 = \{ Z \in M_2(\mathbb{C}) : {}^t Z = Z, \ \mathrm{Im}(Z) > 0 \}$$

$Sp_4(\mathbb{R})$ acts on $\mathfrak{h}_2$ as $\quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$g \cdot Z = (aZ+b)(cZ+d)^{-1}.$$

$$\Gamma_2 = Sp_4(\mathbb{Z})$$

$f : \mathfrak{h}_2 \to \mathbb{C}$, holomorphic, $\quad f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot Z) = \det(cZ+d)^k f(Z)$.

is a Siegel modular form.

$$f(Z) = \sum_t a(t) e^{2\pi i \, \mathrm{Tr}(tZ)}$$

Where the sum is over 2×2 real symmetric matrices that are integral along the diagonal, but can be half-integral off the diagonal

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p3

We would like to know when $\{A(t)\}_t$ gives a Siegel modular form.

$$(1) \quad \left\{ \underbrace{\left(\begin{array}{c|c} {}^t a & \\ \hline & a^{-1} \end{array}\right)}_{m(a)} : a \in GL_2(\mathbb{Z}) \right\}$$

$$(2) \quad \left\{ \left(\begin{array}{c|c} 1_2 & X \\ \hline & 1_2 \end{array}\right) : X \in M_2(\mathbb{Z}),\ {}^tX = X \right\}$$

$$(3) \quad J = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right)$$

Suppose there is a positive real number α so that

- $|A(t)| = O(\det(t)^{\alpha})$
- $f(z) = \sum_{t>0} A(t) e^{2\pi i \, tr(tz)}$ ← This gives transformation under (2) above.

$f(m(a)z) = f({}^t a z a)$

$\iff \quad A({}^t a t a) = A(t).$ Assume our sequence also satisfies this.

We now only need to worry about J.

Let

$$\mathcal{S} = \{ w \in M_2(\mathbb{R}) : \det(w) = 1,\ {}^t w = w,\ w > 0 \}$$

Then for $w \in \mathcal{S}$, $itw \in \mathfrak{H}_2$.

$$f(x,w) = f(i x^{1/2} w) \qquad x \in \mathbb{R}^{\times}_{>0}, \; w \in \mathcal{S}$$

$$\tilde{f}(s,w) = \int_0^\infty f(x,w)\, x^s \frac{dx}{x}.$$

$$\mathfrak{H}_1 = SL_2(\mathbb{R})/SO(2)(\mathbb{R}) \longrightarrow \mathcal{S}$$

$$g \longmapsto g\,{}^t g$$

<u>Fact:</u> $\tilde{f}(s,w) \in L^2(\Gamma\backslash\mathfrak{H}_1)$

We can define a pairing

$$\langle \tilde{f}(s,-), \varphi \rangle = \iint_{\Gamma\backslash\mathfrak{H}_1} \tilde{f}(s,z)\,\varphi(z)\,\frac{dx\,dy}{y^2}$$

$$\langle \tilde{f}(s,-), \varphi \rangle = \Phi(f,\varphi;s)$$

<u>Theorem (Imai):</u> $\Phi(f,\varphi;s) = 2(2\pi)^{-s}\sqrt{\pi}\;\Gamma(s-a_\varphi)\cdot\Gamma(s-b_\varphi)\sum_{t>0}\frac{A(t)}{E(t)}\det(t)^{-s}\cdot\overline{\varphi(Z_t)}$

$$E(t) = \#\{g \in GL_2(\mathbb{Z}) : {}^t g\, t\, g = t\}$$

Theorem (Ibukiyama): $f(Z) = \sum_{t>0} A(t) e^{2\pi i \, Tr(tZ)}$. assume

- $A(t) = A({}^t a t a) \quad \forall \quad a \in GL_2(\mathbb{Z})$
- $|A(t)| = O(\det(t)^\alpha) \quad \alpha > 0$

The following are equivalent

(1) $f(-Z^{-1}) = \det(Z)^k f(Z)$

(2) $\Phi(f, \varphi, k-s) = (-1)^k \Phi(f, \varphi, s)$

for all eigenforms φ.

If either is true, she proves they have analytic continuation.

Let π be a rep. of $GSp_4(\mathbb{A})$, $\pi = \otimes_v \pi_v$.

$L(s, \pi)$

$L(s, \pi \otimes \sigma)$

Theorem:

$$I(s) = \int_{Z(\mathbb{A}) G^+(F) \backslash G^+(\mathbb{A})} E(s,g) \varphi(g) \Theta_\phi(\nu^{-1})(g) \, dg$$

$$= \prod_v \int_{N(F_v) \backslash G_1(F_v)} f_v(s, g_v) \, \varphi_T^{T,v}(g_v) W_v(g, 1) \, dg_v$$

$$= L^S(s, \pi \otimes \chi_T) \prod_{v \in S} \int_{N(F_v) \backslash G_1(F_v)} f_v(s, g_v) \, \varphi_T^{T,v}(g_v) W_v(g, 1) \, dg_v$$

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PS6

$\varphi \in V_\pi$

$$\nu : GSO(2,1)(F) \backslash GSO(2,1)(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

$$E(s,g) = \sum_{\gamma \in P(F) \backslash G(F)} f(s, \gamma g) \qquad f(s,g) \in \mathrm{Ind}$$