L-functions, Modularity, and Functoriality:

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L-functions:

1) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}$ $\quad Re(s) > 1$

- mero cont to $\mathbb{C}$, pole at $s=1$
- $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = Z(1-s)$
- Non-vanishing on $Re(s)=1 \Rightarrow$ P.N.T.
- zeros on $Re(s) = 1/2 \Leftrightarrow$ R.H.

2) $L(s,\chi)$ $\quad \chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}$

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1-\chi(p)p^{-s})^{-1}$$

- $Re(s) > 1$
- entire
- F.E.
- $L(1,\chi) \neq 0 \Rightarrow$ infinite # of primes in arithmetic progressions

1920's Hamburg:

① Artin

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$$

for each p; $\quad L_p(s,\rho) = \det(1-\rho(Frob_p)p^{-s})^{-1}$

$$L(s,\rho) = \prod_p L_p(s,\rho) \qquad Re(s) > 1$$

First L-function that begins life as an Euler product

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· Brauer: $L(s,\rho)$ has mero. cont.

$$\Lambda(s,\rho) = \Gamma(s,\rho) L(s,\rho) = W \Lambda(1-s, \tilde{\rho})$$

Conj: If ρ irred. then $L(s,\rho)$ entire.

Thinking about these L-functions led to the Artin Reciprocity law.

· Arithmetic / geom. object M;

p $\quad L_p(s,M)$ local Euler factor (local information)

$$L(s,M) = \prod_p L_p(s,M) \qquad Re(s) > 1$$

Conj: continuation (entire), $\Lambda(s,M) = \varepsilon(s,M)\Gamma(s,M)L(s,M)$ should have functional equation.

$M = E$ elliptic curve $\quad L(1/2, E)$ BSD ..

M $\quad L(1/2, M)$ Bloch-Kato.

Paradigm:

- $L(s,M)$ defined by E.P.
- Connections to arithmetic built in.
- Analytic properties are hard and mysterious.

② <u>Hecke:</u> Modular forms

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$\mathfrak{H} = \{ z = x + iy : y > 0 \}$

$f : \mathfrak{H} \longrightarrow \mathbb{C}$ holo, holo at cusps.

$\Gamma = SL_2(\mathbb{Z}) \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

$\Gamma \ni \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow f(z)$ has Fourier expansion

$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$, $\quad f$ cusp form means $a_0 = 0$.

$f(z) \rightsquigarrow L(s,f) = \sum \frac{a_n}{n^s} \qquad Re(s) > k/2 + 1$

$\Lambda(s,f) = (2\pi)^{-s} \Gamma(s) L(s,f) = \int_0^\infty f(iy) y^s d^\times y$ (Mellin transform)

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma \qquad z \mapsto -1/z$

<u>Thm:</u> $\Lambda(s,f)$ is nice

- entire continuation
- bounded in vertical strips (BVS)
- $\Lambda(s,f) = \varepsilon(s) \Lambda(k-s,f)$ f.e.

Since $SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$ & Mellin inversion

gives:

Hecke's Converse Theorem: If $L(s)=\sum \frac{a_n}{n^s}$ converges for

$Re(s) > *$ and is nice (entire continuation, BVS, F.E) then

$$f(z)=\sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is modular.

Hecke introduced the Hecke operators $\{T_n\}$.

Thm: If f is an eigenfunction for all T_n, i.e, $T_n f = \lambda_n f$

then $$L(s,f) = \prod_p (1-a_p p^{-s}+ p^{k-1-2s})^{-1}.$$

Analytic Paradigm:

$L(s,f)$

- $Re(s) > *$
- Analytic properties are straight-forward & characterize L-functions of modular forms
- Euler product is mysterious
- arithmetic content is mysterious

Weil's Converse Thm: Characterize $f \in S_k(\Gamma_0(N))$ by

$$L(s,f,\chi)=\sum_{n\geq 1}\frac{a_n \chi(n)}{n^s} \quad \text{with } \chi: (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}$$

Dirichlet characters for all χ w/ $cond(\chi)=M$ relatively prime to N.

Automorphic forms since 1960's (analytic theory):

- $\mathfrak{h} = PGL_2^+(\mathbb{R}) / PO^+(2)$

- Tate's Thesis: adelic theory $\mathbb{A} = \mathbb{R}^\times \times \prod_p{}' \mathbb{Q}_p \underset{\text{disc}}{\supseteq} \mathbb{Q}$

  GL_1 $\qquad$ $\mathbb{A}/\mathbb{Q}$ compact

$$GL_n(\mathbb{A}) = GL_n(\mathbb{R}) \times \prod_p{}' GL_n(\mathbb{Q}_p) \underset{\text{discrete}}{\supseteq} GL_n(\mathbb{Q})$$

$GL_n(\mathbb{A}) / GL_n(\mathbb{Q})$ finite volume modulo center

$\mathcal{A}_0(GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}), \omega) \circlearrowleft GL_n(\mathbb{A})$ right translation

$\quad \parallel$

$\bigoplus m(\pi) V_\pi \qquad (\pi, V_\pi)$ cuspidal auto. reps.

Each $\pi = \pi_\infty \otimes \otimes' \pi_p$ where π_v irred. rep. of $GL_n(\mathbb{Q}_v)$.

To each $\pi_\infty \longmapsto L(s, \pi_\infty) = \Gamma_{\pi_\infty}(s)$

$\pi_p \longmapsto L(s, \pi_p)$ Euler factor of degree n.

$\pi \longmapsto \Lambda(s, \pi) = L(s, \pi_\infty) \prod_p L(s, \pi_p)$.

Following Hecke: $L(s, \pi)$ are nice: $\operatorname{Re}(s) > 1$, entire cont.

$$L(s, \pi) = \varepsilon(s, \pi) L(1-s, \tilde{\pi})$$

Converse Theorem: Given π rep. of $GL_n(\mathbb{A})$, form $L(s,\pi)$. If $L(s,\pi\times\tau)$ is nice for cuspidal rep. τ of $GL_m(\mathbb{A})$, $m<n$, then π is automorphic.

Modularity:

Langlands Conjecture: There should be an injection

$$\left\{ \begin{array}{c} \rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C}) \\ \text{irred} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{cusp. auto.} \\ \text{rep of } GL_n(\mathbb{A}) \\ \pi \end{array} \right\}$$

$$L(s,\rho) = L(s,\pi).$$

Local Langlands: There should be an injection

$$\left\{ \rho_v : Gal(\overline{\mathbb{Q}}_v/\mathbb{Q}_v) \longrightarrow GL_n(\mathbb{C}) \right\} \hookrightarrow \left\{ \begin{array}{c} \pi_v \text{ irred. adm.} \\ \text{rep. of } GL_n(\mathbb{Q}_v) \end{array} \right\}$$

$$L(s,\rho_v) = L(s,\pi_v).$$

This is a theorem!

To get a bijection, one must replace $Gal(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ by Weil-Deligne group W' because the Galois group is "too small".

Global modularity:

$$\rho \qquad \{\rho_v = \rho|_{D_v}\} \longleftrightarrow \left\{\pi_v : \begin{array}{l}\text{rep. of}\\ GL_n(\mathbb{Q}_v)\end{array}\right\}$$

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

$\pi = \otimes \pi_v$

is this auto?

Functoriality:

① Can automorphic forms and L-functions for other groups. H

② "arithmetic parameterization" for auto. rep. of H.

H	GL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}
${}^LH$	$GL_n(\mathbb{C})$	$SO_{2n+1}(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$SO_{2n}(\mathbb{C})$

So we have local Langlands for H

$$\left\{\begin{array}{c}\phi_v : W_v' \longrightarrow {}^LH \\ \text{adm. hom}\end{array}\right\} \twoheadleftarrow \left\{\begin{array}{c}\pi_v \ \text{irred. adm.}\\ \text{reps } H(\mathbb{Q}_v)\end{array}\right\}$$

finite fibers

$$L(s,\phi_v) = L(s,\pi_v)$$

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Functoriality: If we have a homom. ${}^LH \to GL_N(\mathbb{C})$ we should be able to transfer auto. forms π on $H(\mathbb{A})$ to Π on $GL_N(\mathbb{A})$.

Local picture:

$$\begin{tikzcd}
 & {}^LH \arrow[rr] & & GL_N(\mathbb{C}) & \\
\pi_v \arrow[r] & \phi_v & & [illegible] \arrow[r] & \Pi_v \\
 & & W_v' \arrow[uul] \arrow[uur] & &
\end{tikzcd}$$

π_v rep of $H(\mathbb{Q}_v)$

Π_v irred. adm. rep. of $GL_N(\mathbb{Q}_v)$

$$L(s,\pi_v) = L(s,\phi_v) = L(s,\Phi_v) = L(s,\Pi_v).$$

Global Picture:

$$\begin{tikzcd}
 & {}^LH \arrow[rr, hook] & & GL_N(\mathbb{C}) & \\
\pi = \otimes \pi_v \arrow[r, "\cancel{\pi_v}"] & \phi_v & & \Phi_v \arrow[r, mapsto] & \Pi_v \\
 & & W_v' \arrow[uul] \arrow[uur] & &
\end{tikzcd}$$

$\pi = \otimes\pi_v$ cusp auto form on H

$\Pi_v = \otimes \Pi_v$

irred. adm. rep of $GL_N(\mathbb{A})$

$L(s,\pi)$ - - - - Artin - - - - - - - -> $L(s,\Pi)$

<u>Thm:</u> If H is a quasi-split classical group &

π is "globally generic" cusp. rep. of $H(\mathbb{A})$, then

Π is an auto rep. of $GL_N(\mathbb{A})$.