PANTS
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Introduction to p-adic L-functions

This talk is intended to give an introduction to p-adic L-functions to help give prep. for the plenary address by Kartik Prasanna. The goal will be to keep it as low level and appropriate to graduate students as is possible.

One of the most beautiful and useful connections in mathematics is the interplay between analysis and arithmetic in number theory. The analysis generally comes in the form of L-functions. These are functions in the world of complex analysis that are built from local information from the number theoretic object being studied. Remarkably, these L-functions often give us global information back about the object of interest. (class number formula, BSD, etc.) I don't want to spend time on this as you will probably hear more about this in Kartik's talk.

While the powerful tools of complex analysis can be very useful in studying L-functions, they often are not suitable for getting precise arithmetic information. Let X = # theoretic object of interest (# field, characters, elliptic curve, Galois representation) and $L(s,X)$ the complex L-function. What one would like to be able to do is replace the tools of complex analysis with those of p-adic analysis. To do this, we need a p-adic object to replace $L(s,X)$ but that still gives the appropriate information, i.e. it should agree with $L(s,X)$ for the special values s that give the global information.

We will go through some specific examples in a moment, but let me give a little more general overview: The special values of $L(s,X)$ (i.e., at certain integers) suitably normalized are algebraic. If one can show these values "satisfy enough congruences" (we'll make this precise), then one can show that there is a p-adic L-function $\mathcal{L}_p(s,X)$ that p-adically interpolates the special values, i.e., it essentially agrees with $L(k,X)$ for the special values.

An added benefit of working in the world of p-adic analysis is one can often p-adically deform the object X of interest as well, allowing even more info to be gained. Examples of this would be considering modular forms in Hida families or deformations of Galois representations. We will come back to this if there is time.

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
L(s,X) \arrow[d] & & \mathcal{L}_p(s,X) \arrow[d] \\
L(k,X) \arrow[rr, leftrightarrow] & & \mathcal{L}_p(k,X)
\end{tikzcd}$$

easier to study over here b/c we have p-adic analysis ...

We'll start with some easy examples, but first I want to go through a few defs so I can at least give you an idea of how these objects are constructed.

p-adic Measures:

Let X be a compact topological space and let $C(X, \mathbb{Z}_p)$ be the ring of continuous $\mathbb{Z}_p$-valued functions on X. For B a p-adic ring, a $\mathbb{Z}_p$-linear (not nec. a ring homom) map

$$\mu : C(X, \mathbb{Z}_p) \to B$$

is called a B-valued measure on X. For $f \in C(X, \mathbb{Z}_p)$, the image of f under such a μ is denoted by

$$\int_X f \, d\mu \quad \text{or} \quad \int f(x) \, d\mu(x).$$

The general idea is that p-adic L-functions are constructed by constructing appropriate p-adic measures. We won't go into more details as in general the construction is very difficult and complicated. It is best to just understand this through some examples.

Examples:

① Riemann zeta-function:

Recall the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$.

It is a classical fact that for $k \geq 1$, one has

$$\zeta(1-k) = \frac{-B_k}{k}$$

where the B_k are Bernoulli numbers $\left(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{xe^x}{e^x - 1} \right)$.

The values $\zeta(1-k)$ are rational numbers, so we may wish to look for a p-adic function that takes values at $1-k$ of $-B_k/k$. However, it is not hard to see that $\sum_{n \geq 1} \frac{1}{n^s}$ does not converge p-adically as the terms where $p \mid n$ get arbitrarily large. To ~~remedy this~~, we recall the Euler product

$$\zeta(s) = \prod_{\ell} (1-\ell^{-s})^{-1}.$$

To remove the p-terms, we consider the function

$$(1-p^{-s})\zeta(s)$$

instead and try to approximate this p-adically.

Set

$$\zeta^{(p)}(-k) = (1-p^k)\zeta(-k) \quad \text{for any } k \geq 0.$$

These numbers are all p-integral and one has the following congruences:

For $\sum c_n x^n \in \mathbb{Z}[x]$ which satisfies a congruence of the form

$$\sum c_n x^n \equiv 0 \pmod{p^N} \quad \forall \; x \in \mathbb{Z}_p^{\times}$$

there is a congruence

$$\sum c_k \zeta^{(p)}(-k) \equiv 0 \pmod{p^N}$$

Saying these values fit together nicely p-adically...

These congruences together allow one to conclude there is a $\mathbb{Z}_p$-valued p-adic measure μ ~~on the group~~ $\mathbb{Z}_p^{\times}$ $(=X)$ s.t. $\forall \; k \geq 0$ one has

$$\mu(x^k) = \int_{\mathbb{Z}_p^\times} x^k \, d\mu = (1-p^k)\zeta(-k) = \zeta^{(p)}(-k).$$

We now switch to a little more general setting, but still fairly concrete

② Dirichlet characters:

First we recall the Teichmüller character. Let $\alpha \in \mathbb{Z}_p^\times$. We can reduce α mod p to get $\bar{\alpha} \in \mathbb{F}_p^\times$. We know that $f(\bar{\alpha}) = 0$ for $f(x) = x^{p-1} - 1$ and so we can lift $\bar{\alpha}$ uniquely to a $(p-1)^{st}$ root of unity $\omega(\alpha)$ in $\mathbb{Z}_p^\times$ via Hensel's lemma. ~~The~~ Teichmüller character is defined by

$$\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$$
$$x \longmapsto \omega(x).$$

Let M be a positive integer and consider

$$\mathbb{Z}_{p,M} = \varprojlim \mathbb{Z}/p^n M\mathbb{Z} \simeq \mathbb{Z}_p \times \mathbb{Z}/M\mathbb{Z}$$

and

$$\mathbb{Z}_{p,M}^\times = \varprojlim (\mathbb{Z}/p^n M\mathbb{Z})^\times \simeq \mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$$

We say $\chi : \mathbb{Z}_{p,M}^\times \to \mathbb{C}_p^\times$ is a p-adic character if it is a cont. homom. If M is the smallest integer s.t.

$$\chi : \mathbb{Z}_{p,M}^\times \to \mathbb{C}_p^\times$$

we call M the p'-conductor of χ.

Let ψ be a Dirichlet character of conductor $p^n M$. By viewing $(\mathbb{Z}/p^n M)^\times$ as a quotient of $\mathbb{Z}_{p,M}^\times$ we can view ψ as a p-adic character. We write x_p for $x \in \mathbb{Z}_{p,m}$ projected onto $\mathbb{Z}_p$.

We again have a measure μ, this time on $\mathcal{C}(\mathbb{Z}_p^\times, \mathbb{Z}_p[\psi])$. In particular, define

$$\mathcal{C}(\mathbb{Z}_p^\times, \mathbb{Z}_p[\psi]) \cong \mathcal{C}(\mathbb{Z}_p^\times, \mathbb{Z}) \hat{\otimes} \mathbb{Z}_p[\psi]$$

$$\mathcal{L}_p(\psi, s) = \mathcal{L}_p(\psi\chi_s) = \int_{\mathbb{Z}_{p,m}^\times} \psi\chi_s(x)\,d\mu \qquad (s \in \mathbb{Z}_p)$$

where

$$\chi_s(x) = x_p^s\,\omega^{-s}(x). \qquad (s \in \mathbb{Z}_p)$$

This is the Kubota-Leopoldt p-adic L-function. It satisfies the ~~interpolation~~ property that

$$\mathcal{L}_p(\psi\chi_{\cancel{1+k}}) = (1-\psi(p)p^{-k})\,L(-k,\psi) \qquad k \geq 1.$$

Note it is clearer here that one can view the p-adic L-function as a function on characters, though the value we are varying here is just $s \in \mathbb{Z}_p$. This is a 1-variable p-adic L-function. Our next example will be a 2-variable p-adic L-function.

③ Grossencharacters of imaginary quadratic fields:

This may not seem like the next logical step, but these ~~two variable~~ p-adic ~~L-functions~~ will be needed in the following plenary address by Prasanna.

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let $\mathfrak{n} \subset \mathcal{O}_K$ be an ideal. Let $(a,b) \in \mathbb{Z}^2$. A Grossencharacter χ of conductor $\mathfrak{n}$ (not nec. primitive) and type (a,b) is a $\mathbb{C}$-valued multiplicative function on the group of fractional ideals of K prime to $\mathfrak{n}$ (i.e., on $I_K^{\mathfrak{n}}$) s.t. if $\alpha \in K$, $\alpha \equiv 1 \bmod \mathfrak{n}$ then

$$\chi((\alpha)) = \alpha^a \bar{\alpha}^b .$$

We define a character on $(\mathcal{O}_K/\mathfrak{n})^\times$ by

$$\psi_\chi(\alpha) = \chi((\alpha))/\alpha^a \bar{\alpha}^b .$$

We extend this by 0 to get a character on $(\mathcal{O}_K/\mathfrak{n})$.

The L-function attached to χ is given by

$$L(s,\chi) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ \mathfrak{a} \text{ prime} \\ \text{to } \mathfrak{n}}} \frac{\chi(\mathfrak{a})}{\mathrm{Nm}(\mathfrak{a})^s} = \prod_{\mathfrak{p} \nmid \mathfrak{n}} \left(1 - \chi(\mathfrak{p})\mathrm{Nm}\mathfrak{p}^{-s}\right)^{-1} .$$

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be a set of ideal reps. for the ideal class group of K. We choose them to be relatively prime to $\mathfrak{n}$. One has that $\mathfrak{a}_1^{-1}, \ldots, \mathfrak{a}_h^{-1}$ is also a set of reps and so any $\mathfrak{a} \subseteq \mathcal{O}_K$ can be written as

$$\mathfrak{a} = m_i \mathfrak{a}_i^{-1}$$

for some unique i and $m_i \in \mathfrak{a}_i$ determined uniquely up to a unit. We can thus write:

$$L(s,\chi) = \frac{1}{\#\text{ units in } \mathcal{O}_K} \sum_{i=1}^{h} \frac{\chi(\mathfrak{a}_i)^{-1}}{(Nm(\mathfrak{a}_i))^{-s}} \sum_{\substack{m \in \mathfrak{a}_i \\ m \neq 0}} \frac{m^a \bar{m}^b \psi_\chi(m)}{Nm(m)^s}.$$

The term we p-adically approximate is the sum

$$(*) \qquad \sum_{\substack{m \in \mathfrak{a}_i \\ m \neq 0}} \frac{m^a \bar{m}^b \psi_\chi(m)}{Nm(m)^s}.$$

Write

$$\tau_{a,b}(x,y) = x^a y^b.$$

Suppose that p split in K, i.e., $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. We assume our $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ are prime to $\mathfrak{p}$ as well. Suppose that $p \mid \mathfrak{n}$ (this just amounts to discarding the $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ Euler factors. so not surprising we do this.) Decompose the character ψ_χ on $(\mathcal{O}_K/\mathfrak{n})^\times$ into

$$\psi_\chi(m) = \varepsilon_1(m)\,\varepsilon_2(\bar{m})\,g_\chi(m)$$

where ε_1 is the $\mathfrak{p}$-component, ε_2 the $\bar{\mathfrak{p}}$-component, and g_χ the prime to p component. Thus, $(*)$ becomes

$$(*) = \sum_{\substack{m \in \mathfrak{a}_i \\ m \neq 0}} \frac{m^a \bar{m}^b \varepsilon_1(m)\varepsilon_2(\bar{m}) g_\chi(m)}{Nm(m)^s}$$

$$= \sum_{\substack{m \in \mathfrak{a}_i \\ m \neq 0}} \frac{\tau_{a,b}(m,\bar{m})\,\varepsilon_1(m)\varepsilon_2(\bar{m})\, g_\chi(m)}{Nm(m)^s}.$$

Katz constructed a 2-variable p-adic L-function $\mathcal{L}_p(\varepsilon_1^{-1}\varepsilon_2\,\tau_{1-a,b}, g_\chi)(\mathfrak{a}_i)$

that p-adically interpolate the special values of $(*)$ in the quadrant

$$b \geq 0, \quad a \leq -1 \quad \text{and} \quad s = 0.$$

Note that here we are varying the characters, the value of s is fixed! (Also keep in mind, you don't just plug $s=0$ into $(*)$ since this is not part of the original half-plane of convergence!)

④ Other Examples:

There are general conjectures for a vast range of objects that should have p-adic L-functions. Some examples where we know they exist:

- Elliptic curves (wt 2 modular forms, so see next one)
- Modular forms 1: Let f be a wt k newform, char ε.

  Let $\alpha, \bar{\alpha}$ be the roots of $x^2 \pm a_f(p)\, x + \varepsilon(p)\, p^{k-1}$

  (way to pick which is α, which is $\bar{\alpha}$. for future .. assume this is done. depends on if $p \nmid N$ or $p \mid N$, etc)

  Let

  $$\chi(x) = x_p^j\, \psi(x) \qquad 0 \leq j \leq k-2$$

  as above. There is again an appropriate measure $\mu_{f,\alpha}$ and we get a p-adic L-function

  $$\mathcal{L}_p(f, \alpha, \chi, s) := \mathcal{L}_p(f, \alpha, \chi \chi_s).$$

  It p-adically interpolates:

$$\mathcal{L}_p(f, \alpha, \chi) = (*)\ L(f \otimes \overline{\psi}, j+1).$$

Again, we see the variable is $\chi\chi_s$, i.e., the character.

Modular forms 2: You can also vary the modular form in a Hida family $\mathcal{F} \in \Lambda = \mathcal{O}[\![T]\!]$. In this way you are actually varying the modular form (through the Hida family, i.e., through different weights.) Can get 2-variables by varying character as well as varying in the Hida family.