Ho
12-2-12

Ranks of elliptic curves over quadratic extensions:

joint w/ Manjul Bhargava.

Ternary cubic forms

Polynomial in three variables of degree 3. Such a polynomial cuts out a genus one curve in the projective plane.
We work over $\mathbb{Q}$ for this talk. Let $f(x,y,z)$ be such a form w/ coeffs $b_1, \ldots, b_{10} \in \mathbb{Q}$.
$g \in GL_3(\mathbb{Q})$ acts on f by

$$g \cdot f(x,y,z) = f((x,y,z)g)\det(g)^{-1}.$$

This gives an action of $PGL_3(\mathbb{Q})$ on ternary cubic forms.

Parameterization:

$$\left\{ \begin{matrix} \text{ternary cubics} \\ \Delta \neq 0 \end{matrix} \right\} \Big/ \mathbb{Q}^{\times} \times PGL_3(\mathbb{Q}) \xleftrightarrow{1-1} \left\{ \begin{matrix} C & \text{smooth genus one curve} \\ L & \text{deg 3 line bundle on } C \end{matrix} \right\} \Big/ \simeq \qquad (*)$$

Invariant theory:

ternary cubics have 2 invariants S, T of degree 4 and 6.
On the RHS of $(*)$ $\mathrm{Jac}(C): y^2 = x^3 + Sx + T$.

Application: Ranks of E.C.s.

E elliptic curve $/\mathbb{Q}$.
$E(\mathbb{Q})$ f.g. abelian group.
$\mathrm{rk}(E(\mathbb{Q})) = \mathbb{Z}$-rank of $E(\mathbb{Q})$.

Ho
12-2-12
pg2

It is conjectured that half of elliptic curves have rank 1 and half have rank 0.

<u>p-Selmer group:</u>

$$\mathrm{Sel}_p(E) = \left\{ \begin{matrix} C \text{ genus } 1 \text{ w/ } \mathrm{Jac}(C) = E, \text{ deg } p \text{ line bundle} \\ \text{on } C,\ C \text{ locally solvable} \end{matrix} \right\} / \simeq$$

$$\mathrm{rk}_p \, \mathrm{Sel}_p(E) \geq \mathrm{rk}\, E(\mathbb{Q}).$$

$$\begin{tikzcd}
 & \{\text{ternary cubics up to equiv.}\} \arrow[d, leftrightarrow, "1-1"] \\
\left\{\begin{matrix}\text{Same w/} \\ C \text{ loc.} \\ \text{Soluble}\end{matrix}\right\} \arrow[r, leftrightarrow] \arrow[dr, "\mathrm{Sel}_3(E)", "\text{fiber of } E ="'] & \left\{ \begin{matrix} E = \mathrm{Jac}(C) \\ (C, L) \end{matrix} : \begin{matrix} \text{genus } 1 - C \\ \text{deg } 3\ L \end{matrix} \right\} \arrow[d, "\text{take the Jac of } C"] \\
 & \{E \text{ elliptic curves}\}
\end{tikzcd}$$

average # of 3-Selmer elts

$$\underset{x \to \infty}{\approx_{/\lim}} \frac{\#\{\text{ternary cubics}^{\swarrow \text{up to ht } X}\} + \text{sieve}\}}{\#\{\text{ell. curves up to ht } X\}}$$

<u>Thm (Bhargava - Shankar (2010))</u>: The average size of the 3-Selmer group of all ECS is 4.

<u>Cor:</u> The average rank is bounded by $7/6$.

Use p-parity conj:

Thm (Dokchitser²): For $E/\mathbb{Q}$, root # $W(E) = (-1)^{\mathrm{rk}_p \mathrm{Sel}_p(E)}$.

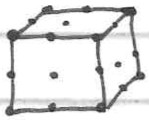
Build pos. density family of ECs w/ equidistributed root #,

Cor: There is a positive proportion of ECs with rank 0.

How to generalize?

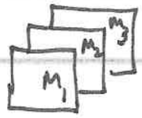
Find other (G, V) related to genus 1 curves.

Rubik's cubes: $(3 \otimes 3 \otimes 3)$



27 numbers. (elt of V)

Call this 27-dimensional space V. We have an action of GL_3^3 on V.



$\rightsquigarrow$ $\det(M_1 x_1 + M_2 x_2 + M_3 x_3)$

ternary cubic f_1.

via symmetry also obtain f_2, f_3.

We obtain three genus 1 curves C_1, C_2, C_3 and they are all isomorphic. ~~$C_1 \to C_2$ $[x_1 : x_2 : x_3]$~~

ex: $C_1 \longrightarrow C_2$

$[x_1 : x_2 : x_3] \in C_1 \subseteq \mathbb{P}^2$

$\det(\underbrace{M_1 x_1 + M_2 x_2 + M_3 x_3}_{\text{ker of this matrix gives pt on } C_2.}) = 0$

$$\begin{tikzcd} & C_1 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ C_3 \arrow[rr, leftrightarrow] & & C_2 \end{tikzcd}$$

not commutative!

P = point on $\mathrm{Jac}(C_1)$ that gives the automorphism going around the diagram in the clockwise direction.

<u>Thm:</u>

$$\begin{tikzcd} \text{nondeg } V(\mathbb{Q}) / G(\mathbb{Q}) \arrow[r, leftrightarrow, "1\text{-}1"] & \left\{ \begin{array}{l} (C, L_1, L_2, L_3) \\ L_1^{\otimes 2} \cong L_2 \otimes L_3,\ L_1 \neq L_2, L_3 \end{array} \right\} \arrow[d, leftrightarrow] \\ & \left\{ (C, L, P),\ 0 \neq P \in \mathrm{Jac}(C) \right\} \end{tikzcd}$$

invariants of degs 6, 9, 12
a_6, a_9, a_{12}

$\mathrm{Jac}(C): y^2 + a_9 y = x^3 + a_6 x^2 + a_{12} x.$

$P = (0,0)$

$\mathcal{F}_1$ = family of elliptic curves of this form

~~$\{\text{Rubik's cubes}\}/\sim \longleftrightarrow$~~

We obtain a similar pic to above

Ho
12-2-12
p55

$$\begin{tikzcd}
 & \{\text{rubiks cube}\} \arrow[d, leftrightarrow] \\
\left\{\begin{array}{c}\text{same w/}\\ C \text{ loc.}\\ \text{soluble}\end{array}\right\} \arrow[r, hook] \arrow[dr, "\text{fiber} = \mathrm{Sel}_3(E)"'] & \left\{\begin{array}{c}(C, L, E = \mathrm{Jac}(C))\\ E \in \mathcal{F}_1\end{array}\right\} \arrow[d] \\
 & \{E \in \mathcal{F}_1\}
\end{tikzcd}$$

Thm: The average size of 3-Selmer group of ECs in $\mathcal{F}_1$ is 12.

Cor: There exist a positive proportion of ECs in $\mathcal{F}_1$ w/ rank 1.

Hermitian matrices wrt $K/\mathbb{Q}$:

$$\begin{pmatrix} a_i & d_i & e_i \\ \bar{d_i} & b_i & f_i \\ \bar{e_i} & \bar{f_i} & c_i \end{pmatrix} \qquad \begin{array}{l} a_i, b_i, c_i \in \mathbb{Q} \\ d_i, e_i, f_i \in K. \end{array}$$

$$V \circlearrowleft GL_3(\mathbb{Q}) \times GL_3(K)$$

$\rightsquigarrow F_1, F_2, F_3$ ternary cubic forms $/K$

$\nearrow$ defined over $\mathbb{Q}$

$$F_2 = \overline{F_3}.$$

def over $\mathbb{Q}$

$$\begin{tikzcd}
& C_1 \arrow[dl] \arrow[dr, leftrightarrow, "\cong/_K"] & \\
C_2 \arrow[rr, leftrightarrow] & & C_3
\end{tikzcd}$$

not. comm.

$E = \mathrm{Jac}(C_1)$ def $/\mathbb{Q}$

$P \in E(K)$ w/ $-P = \overline{P}$ $\Rightarrow$ point P' on $E^K(\mathbb{Q})$

Thm:

$$\begin{tikzcd}
\text{nondeg } V(\mathbb{Q}) / G(\mathbb{Q}) \arrow[r, leftrightarrow] & \vdots \arrow[d, leftrightarrow] \\
& \{(C, L, P),\ 0 \neq P \in \mathrm{Jac}(C)^K\}
\end{tikzcd}$$

$$\begin{tikzcd}
& \{\text{rubik's cubes}\} \arrow[d, leftrightarrow] \\
\left\{\begin{matrix}\text{Same w/} \\ C \text{ loc.} \\ \text{soluble}\end{matrix}\right\} \arrow[r, hook] \arrow[dr] & \{(C, L, E = \mathrm{Jac}(C)) \ E^K \in \mathcal{F}_1 \arrow[d] \\
& \{E^K \in \mathcal{F}_1\}
\end{tikzcd}$$

Thm: The average size of 3-Selmer group of ECs in $\mathcal{F}_1^K$ is 4.

Corl: There exists a pos. proportion of ECs in $\mathcal{F}_1^k$ w/ rk 0.

fact: $\mathrm{rk}\, E(k) = \mathrm{rk}\, E(\mathbb{Q}) + \mathrm{rk}\, E^k(\mathbb{Q})$

Thm: $\exists$ a pos prop. of ECs in $\mathcal{F}_1^k$ w/
$\mathrm{rk}\, E(\mathbb{Q}) = \mathrm{rk}\, E(k) = 1, 2, 3$.

This has applications to Hilbert's 10th problem.