Gross

12-1-12

PS 1

On the values of zeta functions at negative integers and dimensions of spaces of modular forms:

$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$ $Re(s) > 1$

and simple pole at $s \to 1$.

Euler:

at 28 years old: he proved $\zeta(2) = \frac{\pi^2}{6}$.

later $\zeta(2n) = \pi^{2n}$ (rat. #).

at 30: $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ $\leadsto$ $\sum_{p \le x} \frac{1}{p} = \log(\log(x)) + \cdots$

at 32: $\zeta(1-d)$ $d \geq 1$ all rational numbers

$2^d (2^d - 1)\zeta(1-d)$ is an integer.

$d=2$ $\zeta(-1) = -\frac{1}{12}$.

$\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ $S_{2k}(N)^{new}$

Suppose $N = \prod_S p \prod_T q^3$. $T \neq \emptyset$ and contains $q > 3$ or 2 primes q_1, q_2.

Then

$$\dim S_{2k}(N)^{new} = \frac{2k-1}{12} \prod_p (p-1) \prod_T (q-1)(q^2-1)$$

where the ζ-function comes in since $\zeta(-1) = \frac{-1}{12}$.

at 42: $\pi^{-s/2}\Gamma(s/2)\zeta(s) = Z(s)$

Showed $Z(s) = Z(1-s)$ for $s \in \mathbb{Z}$.

Riemann: zeros of $Z(s)$ have $Re(s) = 1/2$?

PNT $\Leftrightarrow$ no zeros with $Re(s) = 0, 1$.

Euler worked with the series

$$\zeta^*(s) = 1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} + \cdots$$
$$= \zeta(s) - 2\,2^{-s}\zeta(s)$$
$$= (1 - 2^{1-s})\,\zeta(s).$$

$$\zeta^*(0) = 1 - 1 + 1 - \cdots$$
$$= 1 - x + x^2 - x^3 + \cdots \Big|_{x=1}$$
$$= \frac{1}{1+x}\Big|_{x=1} \qquad |x| < 1$$

} origins of analytic continuation

$$= 1/2.$$

Then $\zeta(0) = -1/2$.

$$\zeta^*(-1) = 1 - 2 + 3 - 4 + \cdots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \cdots \Big|_{x=1}$$
$$= \frac{d}{dx}\left(x - x^2 + x^3 + \cdots\right)\Big|_{x=1}$$
$$= \frac{d}{dx}\left(\frac{x}{1+x}\right)\Big|_{x=1}$$
$$= \frac{1}{(1+x)^2}\Big|_{x=1}$$
$$= \frac{1}{4}$$

So $\zeta(-1) = -\frac{1}{12}$.

Continuing like this gives

$$\zeta^*(-n) = 1 - 2^n x + 3^n x^2 + \cdots \Big|_{x=1}$$

$$= P_n(x)\big|_{x=1}$$

$$= \frac{d}{dx}\left(x P_n(x)\right)\Big|_{x=1}$$

$$= \frac{R_n(x)}{(1+x)^n}\Big|_{x=1} \qquad R_n(x) \in \mathbb{Z}[x]$$

k = # field or a function field of curve X over a finite field $\mathbb{F}_q$.

$Z(s) = \prod_v Z_v(s)$ $\quad$ $Re(s) > 1$, Extends meromorphically to $\mathbb{C} - \{0,1\}$.
simple poles

v complex place : $Z_v(s) = (2\pi)^{-s}\Gamma(s)$
v real place : $Z_v(s) = (\pi)^{-s/2}\Gamma(s/2)$ $\quad$ poles at $s=1$.
v finite : $Z_v(s) = (1-q_v^{-s})^{-1}$.

$Z(1-s) = A^{\frac{1}{2}-s} Z(s)$ $\qquad$ $A = |d_k|$ $\quad$ # field
$\qquad = q^{2g-2}$ $\quad$ fctn. field.

S finite set of places $\supset S_\infty$, nonempty

$$\zeta_S(s) = \prod_{v \notin S} Z_v(s)$$

Ex: $k = \mathbb{Q}$, $S = \{\infty\}$ then $\zeta_S(s) = \zeta(s)$.

In fctn field case,

$$\zeta_S(s) = Z(s) \underbrace{\prod_{v \in S}(1-q^{-s})}_{\text{regular at } s=0.}$$

still has pole at $s=1$.

T nonempty set of places disjoint from S. (all finite)

$$\zeta_{S,T}(s) = \zeta_S(s) \prod_{v \in T} (1 - q_v^{1-s}) \qquad \text{regular for all } s \in \mathbb{C}.$$

<u>Ex:</u> $S = \{\infty\}$ $T = \{2\}$ $k = \mathbb{Q}$.

$$\zeta_{S,T}(s) = \zeta^*(s) = \zeta(s)(1 - 2^{1-s}).$$

$s = 1$ $\quad 1 - 1/2 + 1/3 - \frac{1}{4} + \cdots = \log 2.$

$k = \mathbb{F}_q(X)$: g = genus of X.

$$Z(s) = \frac{P_{2g}(q^{-s})}{(1-q^{-s})(1-q^{1-s})}, \quad P_{2g} \in \mathbb{Z}[x] \text{ monic deg } 2g.$$

$$\zeta_{S,T}(s) = P_{2g}(q^{-s}) \frac{\prod_{v \in S}(1 - q_v^{-s})}{1 - q^{-s}} \frac{\prod_{v \in T}(1 - q_v^{1-s})}{(1 - q^{1-s})} \qquad q_v = q^{d_v}$$

$$= P^*_{2g-2+\deg S + \deg(T)}(q^{-s}).$$

Then $\zeta_{S,T}(1-d) \in \mathbb{Z}$.

$P^*_{2g-2+\deg S+\deg T}$ ← char poly of Frob on $H^1(X - S; T)$.

functional eqtn doesn't occur here b/c

$$H^1(X-S;T)^\vee \simeq H^1(X-T;S).$$

Special case: $X \simeq \mathbb{P}^1$, $S = \{\infty\}$, $T = \{0\}$, $\deg S = \deg T = 1$.

$P^*(x) = 1$

$\Rightarrow \zeta_{S,T}(s) = 1$ for all s.

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p95

$y^2 = x(x-1)(x-t)$ over $\mathbb{F}_q(t)$ q odd
$L(E/_{\mathbb{F}_q}, s) = 1$

Now for number fields: (tot real.. any complex ~~place~~ makes the values 0)

Thm (P. Cassou-Noguès, Deligne, Ribet): $S' \supset S_\infty$, $T = \{\mathfrak{q}\}$

$$\zeta_{S'}(s)(1-N\mathfrak{q}^{1-s}) = \zeta_{S,T}(s).$$

Then

$$\zeta_{S,T}(1-d) \in \mathbb{Z}[\tfrac{1}{q}] \qquad \text{where } q = \text{res. char } \mathfrak{q}.$$

$T = \{\mathfrak{q}, \mathfrak{p}\}$ diff res. char.

$$\zeta_{S,T} = \zeta_S (1-N\mathfrak{q}^{1-s})(1-N\mathfrak{p}^{1-s}) \in \mathbb{Z}[\tfrac{1}{p}] \cap \mathbb{Z}[\tfrac{1}{q}] = \mathbb{Z}.$$

$$\frac{1}{2^n}\zeta_{S,T}(1-d) \in \mathbb{Z} \qquad d \geq 2.$$

k global
G ~~semisim~~ simple, simply-connected, split algebraic over k
(SL_n, Sp_{2n}, G_2, E_8, ...)
T max. torus
$\dim(T) = \ell$ = rank of G.
for examples: $SL_n \to n-1$
$Sp_{2n} \to n$
$G_2 \to 2$
$E_8 \to 8$

$d_1, \ldots, d_\ell$ = degrees of inv. polys of G on $\mathrm{Lie}(G)$.

- SL_n $\quad 2, 3, 4, \ldots, n$

  Sp_{2n} $\quad 2, 4, 6, \ldots, 2n$

  G_2 $\quad 2, 6$

  E_8 $\quad 2, 8, 12, 14, 18, 20, 24, 30.$

$$\# G(\mathbb{F}_q) = q^N \prod_{i=1}^{l} (q^{d_i} - 1)$$

$$l + 2N = \dim(G)$$

Interpretation of $\prod_{i=1}^{l} \zeta_{S,T}(1-d_i) \in \mathbb{Z}$.

What does this count?

$$G(k) \hookrightarrow G(\mathbb{A}) = \coprod_v G(k_v).$$

discrete $\qquad$ adeles of k., locally compact

$G(k) \backslash G(\mathbb{A})$ has finite volume

$$\int_{G(k)\backslash G(\mathbb{A})} \mu_{Tam} = 1. \qquad \mu_{Tam} = \text{Tamagawa measure.}$$

$$\mu = \prod \mu_v = c \cdot \mu_{Tam} \qquad \int \mu = c.$$

We construct $\mu_{S,T}$. and in this case $c = \prod_{n=1}^{d} \zeta_{S,T}(1-d_i) \cdot (*)$.

Find this in "On the motive of a reductive group".

$$\int_{G(k)\backslash G(\mathbb{A})} \mu_{S,T} = c.$$

Trace formula:

$$\sum_{\substack{\text{with certain local properties}}} \text{mult of } \pi \text{ in } L^2(G(k)\backslash G(\mathbb{A})) = \sum_{\substack{\text{conj. classes } \gamma \\ \text{in } G(k)}} \mathcal{O}_\gamma$$

orbital integrals.

Can pick a suitable test function so

$v \in S \qquad \pi_v = \text{Steinberg}$

$v \in T \qquad \pi_v = \text{simple - supercuspidal}$

$v \notin S \cup T \qquad \pi_v = \text{unram.}$

$\pi = \otimes \pi_v.$

In this particular case, all $\mathcal{O}_\gamma = 0$ unless $\gamma \in Z(k) =$ center.
For $\gamma \in Z(k)$, we get the orbital integral is $\int_{G(k)\backslash G(\mathbb{A})} \mu_{S,T}$.

In the classical case:
The condition at T gives $q^3 \| N$ and at S gives $p \| N$.