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PSI

Selmer ranks of quadratic twists of elliptic curves

$E/_K \qquad y^2 = x^3 + Ax + B$

$E(K)$ rational pts.

Thm (Mordell-Weil): $E(K)$ is a finitely generated abelian group

$$E(K) \cong \mathbb{Z}^r \oplus E(K)_{tors}$$

$r =$ rank of $E(K)$.

Consider the following short exact sequence Galois-modules:

$$0 \to E[2] \to E(\bar{K}) \xrightarrow{[2]} E(\bar{K}) \to 0.$$

This gives rise to a long exact sequence of cohomology:

$$\underset{\substack{\| \\ E(K)}}{H^0(K, E(\bar{K}))} \xrightarrow{[2]} \underset{\substack{\| \\ E(K)}}{H^0(K, E(\bar{K}))} \to H^1(K, E[2]) \to$$

This shows that

$$E(K)/_{2E(K)} \hookrightarrow H^1(K, E[2]).$$

This is very difficult to use because $H^1(K, E[2])$ is too large. Thus, we look at the completions instead:

$$\begin{tikzcd} E(K)/_{2E(K)} \arrow[r, "\delta"] \arrow[d] & H^1(K, E[2]) \arrow[d, "Res_v"] \\ E(K_v)/_{2E(K_v)} \arrow[r, "\delta_v"] & H^1(K_v, E[2]) \end{tikzcd}$$

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Define

$$H^1_f(K_v, E[2]) := \delta_v\left(E(K_v)/2E(K_v)\right) \subseteq H^1(K_v, E[2]).$$

and

$$Sel_2(E/K) = \{ c \in H^1(K, E[2]) : Res_v(c) \in H^1_f(K_v, E[2]) \ \forall \text{ place } v \text{ of } K \}.$$

We have the following short exact sequence

$$0 \to E(K)/2E(K) \to Sel_2(E/K) \to Ш(E/K)[2] \to 0.$$

Define

$$d_2(E/K) = \dim_{\mathbb{F}_2} Sel_2(E/K) - \dim_{\mathbb{F}_2} E(K)[2].$$

With this definition we have

$$rk(E/K) \leq d_2(E/K).$$

Conj. $d_2(E/K) \equiv rk\, E \pmod 2$.

Thm. (Bhargava-Shankar): $Avg(\# Sel_2(E/K)) = 3$

$$Avg(\# Sel_3(E/K)) = 4$$

$$Avg(\# Sel_4(E/K)) = 7$$

$$Avg(\# Sel_5(E/K)) = 6$$

~~Conjecture~~ $\Rightarrow$

$$Avg(rk(E/\mathbb{Q})) \leq 0.99.$$

We will now look at a special family of elliptic curves,

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namely quadratic twists.

Let $d \in K^\times$. $E^d = dy^2 = x^3 + Ax + B$. Set $E = E^1$.

$$E^d \xrightarrow{\sim} E$$
$$(x,y) \longmapsto (x, \sqrt{d}y).$$

This is an isomorphism over $F = K(\sqrt{d})$. We will also denote $E^F := E^d$ because the field is what is important for this curve. For each quadratic extension of K we get a quadratic twist of E.

<u>Goldfeld's Conjecture:</u> 50% of the quadratic twists of E have rank zero, 50% have rank 1, 0% have rank ≥ 2. (Infinitely many have rank ≥ 2, but the density of these is measure 0.)

$$\{\text{quadratic twists of } E/\mathbb{Q}\} \longleftrightarrow \{\text{quadratic exts of } \mathbb{Q}\}$$
$$\updownarrow$$
$$\{\text{sq. free integers}\}$$

$$\lim_{X\to\infty} \frac{\#\{\square\text{-free } d,\ |d|<X,\ \mathrm{rk}(E^d)=0\}}{\#\{\square\text{-free},\ |d|<X\}} = 1/2.$$

Let $S(X) = \{\square\text{-free } d \in \mathbb{Z},\ |d| \leq X\}$.

$N_r(X) = \{d \in S(X) : d_2(E^d/\mathbb{Q}) = r\}$.

$\quad \shortparallel$

$N_r(E, X)$

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Consider

$$\lim_{X\to\infty} \frac{\# N_r(E,X)}{\# S(X)}.$$

<u>Thm (Heath-Brown, Swinnerton-Dyer, Kane):</u> Suppose that $E(\mathbb{Q})[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, E does not have a cyclic 4-isogeny, then

$$\lim_{X\to\infty} \frac{\# N_r(E,X)}{\# S(X)} = \alpha_r$$

If one sums over α_r for r even one gets $1/2$, sum over α_r for odd r gives $1/2$ as well.

$\alpha_0 \approx 0.21$

$\alpha_1 \approx 0.42$

There are two other possibilities for $E(\mathbb{Q})[2]$:

$$E(\mathbb{Q})[2] = 0$$
$$E(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z}.$$

<u>Thm (K., Rubin, Mazur):</u> Assume $E(\mathbb{Q})[2]=0$, $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \simeq S_3$.

for every $m \in \mathbb{Z}^+$, $X \in \mathbb{R}^+$, $\exists\, B_m(X) \subseteq S(X)$

$$\bigcup_{m,X} B_m(X) = \{\square\text{-free integers}\}$$

$$\lim_{m\to\infty} \lim_{X\to\infty} \frac{\#\{d \in B_m(X) : d_2(E^d/\mathbb{Q}) = r\}}{\# B_m(X)} = \alpha_r.$$

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<u>Thm (K.):</u> assume $E(\mathbb{Q})[2] \simeq \mathbb{Z}/2\mathbb{Z}$, E does not have a cyclic 4-isogeny over $\mathbb{Q}(E[2])$. Then we have
fix r:

$$\liminf_{X\to\infty} \frac{\# N_{\geq r}(E, X)}{\# S(X)} \geq 1/2.$$