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Higher Theta Functions:

1. Introduction:

Jacobi theta function: $\Theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$, $z \in \mathfrak{H} = \{x+iy,\ y>0\}$

$$= \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i m z}$$

$$a(m) = \begin{cases} 2 & m = \square,\ m \neq 0 \\ 1 & m = 0 \\ 0 & o/w. \end{cases}$$

· Used by Riemann in his 2nd proof of f.e. of $\zeta(s)$.

Transformation: If $\gamma \in \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right\}$, $c \in \mathbb{Z}$

then $\Theta(\gamma z) = j(\gamma, z)\Theta(z)$ where $\gamma z = \frac{az+b}{cz+d}$

with $j(\gamma, z) = \begin{cases} \varepsilon_d^{-1} \left(\frac{c}{d}\right)(cz+d)^{1/2} & c \neq 0 \\ 1 & c = 0 \end{cases}$

where $\mathrm{Re}(cz+d)^{1/2} > 0$ and

$$\varepsilon_d^{-1} = \begin{cases} 1 & d \equiv 1\ (4) \\ i & d \equiv 3\ (4) \end{cases}$$

and $\left(\frac{c}{d}\right)$ is quadratic Kronecker symbol

Consequences: study powers of $\Theta(z)$, use theory of modular forms to give estimates on # of ways N = sum of squares.

Also, from the associative property

$$j(\gamma_1 \gamma_2, z) = j(\gamma_1, \gamma_2 z)\, j(\gamma_2, z).$$

2: Analogue for # fields:

Replace $\mathbb{Q}$ by K and $\mathbb{Z}$ by $\mathcal{O}_K$.

Done in general by Hecke.

Concrete example: $K = \mathbb{Q}(i)$

Want an action of $SL_2(\mathbb{Z}[i])$ ... doesn't acts on $\mathfrak{H}$.

Instead, $SL_2(\mathbb{C})$ (and its subgroups of course) act on

$$\mathfrak{H}^3 = \{ x + yk \text{ quaternions} \mid x \in \mathbb{C},\ y \in \mathbb{R}_{>0} \}.$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az+b)(cz+d)^{-1}.$$

$$\Theta_{\mathbb{Q}(i)}(z) = y^{1/2} \sum_{\alpha \in (1+i)^{-1}\mathbb{Z}[i]} e^{\pi i (x\alpha^2 + \bar{x}\bar{\alpha}^2 + 2iy|\alpha|^2)}.$$

Then for $\gamma \in \Gamma((1+i)^3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[i]) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod (1+i)^3 \right\}.$

$$\Theta_{\mathbb{Q}(i)}(\gamma z) = \kappa_2(\gamma)\, \Theta(z)$$

where

$$\kappa_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{c}{d}\right)_2 & c \neq 0 \\ 1 & c = 0 \end{cases}$$

where $\left(\frac{c}{d}\right)_2$ is the quadratic residue symbol in $\mathbb{Q}(i)$.

Consequence: the map

$$\kappa_2 : \Gamma((1+i)^3) \to \{\pm 1\}$$

is a homomorphism.

3. Kubota symbol:

Thm (Kubota 1966): Let K be a # field s.t. K contains n n^{th} roots of unity. Let
($n>2$)

$$\kappa_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{c}{d}\right)_n & c \neq 0 \\ 1 & c = 0 \end{cases}$$

where $\left(\frac{\cdot}{\cdot}\right)_n$ is the n^{th} power residue symbol.
($\left(\frac{\cdot}{\cdot}\right)_n$ is an n^{th} root of unity) Then $\exists$ ideal I of $\mathcal{O}_K$ s.t.

$$\kappa_n : \Gamma(I) \subseteq SL_2(\mathcal{O}_K) \longrightarrow \mu_n = \{ n^{th} \text{ roots of } 1 \}$$

is a homomorphism.

Pf based on n^{th} power reciprocity.
Ivanov (2011): results on level.

Question: Is there an analogue of $\Theta(z)$ and $\Theta_{\mathbb{Q}(i)}(z)$ transforming under κ_n, $n>2$? If so, what are its Fourier coefficients?

Here K is purely imaginary since $n>2$ so we are looking for a function on $(\mathfrak{h}^3)^{r_2}$ where r_2 is the number of pairs of complex embeddings of K.

The objects will be the
"higher theta functions."

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4 Higher theta functions:

Back to Jacobi case:

Question: If we know $j(\gamma, z)$ is a multiplier system, can we recover $\Theta(z)$?

Answer: It is the residue of an Eisenstein series constructed using $j(\gamma, z)$.

Let
$$E_2(z,s) = \zeta(2s) \sum_{\gamma \in (\pm(\begin{smallmatrix}1 & * \\ 0 & 1\end{smallmatrix}) \backslash \Gamma_0(4))} j(\gamma,z)^{-1} \operatorname{Im}(\gamma z)^{s/2} \quad (\operatorname{Re}(s) >> 0)$$

Since
$$E_2(z+1,s) = E_2(z,s)$$
it has a Fourier expansion:
$$E_2(z,s) = \sum_{m \in \mathbb{Z}} a(m,s)\, W_{\operatorname{sgn}(m)}(|m|y, s)\, e^{2\pi i m x}.$$

<u>Fact:</u> ($\approx$ Maass, 1937) If $m = m_0 m_1^2$ and m_0 is square-free, then
$$a(m,s) = L(s, \chi_{m_0})\, b(s, m_0, m_1)$$
where
$$b(s, m_0, m_1) = \prod_{p \mid m_1} (\text{finite polys in } p^{-s})$$

Here $L(s, \chi_{m_0})$ is a quadratic Dirichlet L-fctn,
$$\chi_{m_0}(a) = \left(\frac{m_0}{a}\right).$$

First pole of $E_2(z,s)$ is at $s=1$. (terms with $m_0 = 1$)
residue is $c\,\Theta(z)$ where c is a non-zero constant.

We now play the same game for higher theta functions.

Let $\quad E_n(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(I)} \kappa_n(\gamma)\, \mathrm{Im}(\gamma z)^s, \quad \mathrm{Re}(s) \gg 0.$

$z \in (\mathfrak{h}^3)^{r_2}$, $\mathrm{Im}(z) = y_1 \cdots y_{r_2}$.

__Fact:__ (Selberg / Kubota): E_n has analytic continuation and functional equation. Pole at $s = \frac{1}{2} + \frac{1}{n}$.

__Def:__ $\quad \Theta_n(z) = \mathrm{Res}_{s=\frac{1}{2}+\frac{1}{n}} E_n(z,s)$

What are its Fourier coefficients?

__First approach:__ • Hecke theory ... $T(p^n)$ where p is a prime
This gives some coefficients, but not all because there are not enough Hecke operators.

- Periodicity (Kazhdan-Patterson): relate coeff. at mp^n to coefficient at m.
- $n=3$: Patterson determined coefficients. Interesting ones are cubic Gauss sums
- $n=4$: there is one unknown coeff. for each prime. It is expected to be related to 4^{th} order Gauss sum.
- $n>4$: New conjecture (Shinta, F., Hoffstein, 2012): for coeff. $\tau^{(6)}$ of 6^{th} fold Theta function

$$\sum \frac{\tau^{(6)}(m^2)}{m^2} = \sum \frac{g^{(3)}(d)}{d^4} \sum \frac{\tau^{(2)}(m)}{m^4}$$

$g^{(3)}$ cubic Gauss sums

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- Proved for rational functions field
- Compatible with Hecke
- poles, Γ- functions match.