Families of Automorphic L-functions

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Outline

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(One could choose T differently.) This can be enhanced to

Ex 2. projective system over $T = \mathbb{Z}_{\geq 1}$

$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1.$

Ex 3. vector bundle $X \to T$

$=$ family of vec spaces $\{X_t\}_{t \in T}$ over a manifold or an alg variety T

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The following are two special cases of Ex 4:

Ex 5. Legendre family of elliptic curves (say $T = \mathbb{A}^1_{\mathbb{C}} \backslash \{0, 1\}$)

$=$ family of curves $y^2 = x(x-1)(x-t)$, $t \in \mathbb{C} \backslash \{0, 1\}$.

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Ex 6. reductions of algebraic variety X over $T = \mathrm{Spec} \mathbb{Z}$ or $T = \mathrm{Spec} \mathbb{Z}[1/S]$

$= \{X \mod p\}_{p:\mathrm{prime}}$ (+ gen fiber over $\mathbb{Q}$).

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3. $\mathrm{inv}(X_p) = \#X(\mathbb{F}_p)$, where $(p) \in \mathrm{Spec}\mathbb{Z} = T$.

Dist problem for families (2) - over set of primes

Deligne's proof of Weil conj implies:

Proposition

Assume $X \to \mathrm{Spec}\mathbb{Z}[1/S]$ *is smooth and proper.*
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Ex: X = ell curve; say $y^2 = x(x-1)(x-t)$, $t \in \mathbb{Z} \backslash \{0,1\}$
S = set of primes dividing t

$p + 1 - 2\sqrt{p} \leq \#X(\mathbb{F}_p) \leq p + 1 + 2\sqrt{p}$. (Hasse, 1930s)

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- [Q] Dist of $a_2, a_3, a_5, \ldots$ in $[-2, 2]$? $\cdots$ to be revisited!

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   e.g. rank r of ell curve $E/\mathbb{Q}$, $E(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus (\mathrm{fin})$:
   - individual rank - BSD conj,
   - avg rank in families - recent progress (Bhargava-Shankar)

Entering Part 2

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   - Original Sato-Tate conjecture (for elliptic curves)
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♠ variant: use a weighted average.

Equidistribution - when $\mathcal{X}$ is finite

(Example: $Y_t = \{p \le t : p \nmid N\}$, $\mathcal{X} = (\mathbb{Z}/N\mathbb{Z})^\times$, $a_x = 1/\varphi(N)$.)

- $\{Y_t\}_{t \in \mathbb{Z}_{\ge 1}}$ = disc family; invariant $i : Y_t \to \mathcal{X}$,
- $P_x(Y_t) := \#\{y \in Y_t : i(y) = x\}/|Y_t|$ so that

$$\mu^{\mathrm{count}}_{Y_t} := \frac{1}{|Y_t|} \sum_{y \in Y_t} \delta_{i(y)} = \sum_{x \in \mathcal{X}} P_x(Y_t) \delta_x.$$

- A prob measure μ on $\mathcal{X}$ has the form

$$\mu = \sum_{x \in \mathcal{X}} a_x \delta_x, \quad (\sum_{x \in \mathcal{X}} a_x = 1).$$

I said $\{Y_t\}$ is μ-**equidistributed** if $\mu^{\mathrm{count}}_{Y_t} \to \mu$ as $t \to \infty$, which is true iff $P_x(Y_t) \to a_x$ for all $x \in \mathcal{X}$.

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Conjecture (Sato-Tate, 1960s)

$\{a_E(p)\}_{p \leq N}$ *is equidist. on* $[-2, 2]$ *w.r.t* $\mu^{\mathrm{ST}} = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx$.

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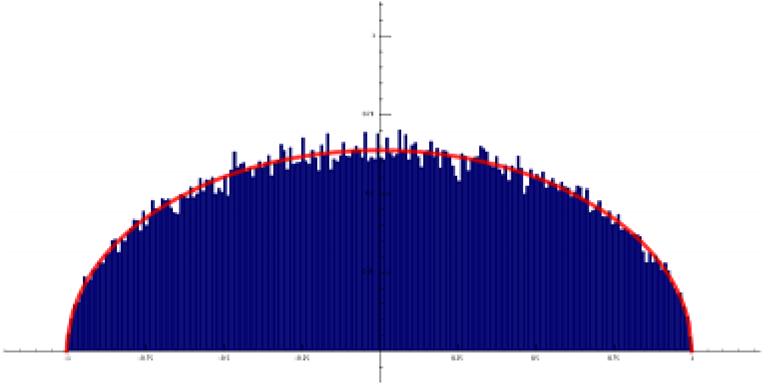
$*$ "Automorphic analogue" also proved for GL_2. (ell curves $\leftrightarrow$ modular forms.) For general algebraic varieties and general automorphic reps, the analog conj is *wide open*.

Sato-Tate conjecture for elliptic curves - graphics



source: Barry Mazur, *Finding meaning in error terms*, 2007.

- red $= \mu^{\mathrm{ST}}$, blue $= \mu^{\mathrm{count}}$ from $\{a_E(p)\}$.

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Examples

- $\pi = \mathbf{1},\ M = \mathrm{Spec}\mathbb{Q},\ \rho = \mathbf{1} \Rightarrow L(s,\pi) = \cdots = \zeta(s)$.

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- f (cuspform) $\overset{\text{Wiles et al}}{\leftrightarrow}$ E (ell. curve) $\leftrightarrow$ Gal rep on $T_l E \otimes \mathbb{Q}_l$.

Automorphic forms and representations - (1)

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Here's a correct but lazy way: auto form of GL_n is

$$f \in L^2(GL_n(\mathbb{Q})\mathbb{R}^\times \backslash GL_n(\mathbb{A})).$$

(Here $\mathbb{A} \approx \prod_p \mathbb{Q}_p \times \mathbb{R}$, $GL_n(\mathbb{A}) \approx \prod_p GL_n(\mathbb{Q}_p) \times \mathbb{R}$.)

An irred constituent of the regular rep of $GL_n(\mathbb{A})$ on $L^2(\cdots)$ is auto rep of $GL_n(\mathbb{A})$.

Problem: Maybe too vague!

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- What are interesting invariants for auto forms/reps?
  (a) $|\mathcal{F}_k|$ ($\approx$ genus of $\Gamma\backslash\mathfrak{H}^*$ if $k=2$), may ask about growth.
  (b) T_p-eigenval. (or local inv in L-fcns). $\cdots$ Our concern

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Some nice properties:

1. Analytic continuation: $L(s,\pi)$ extends to all $s \in \mathbb{C}$ (except finitely many poles).
2. Euler product: $L(s,\pi) = \prod_p L_p(s,\pi)$,
   Ex: $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, $\mathrm{Re}(s) > 1$.
3. Functional Equation: $\Lambda(s,\pi) = \Lambda(1-s,\pi^\vee)$
   (Λ = completed L-function).

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♠ In generalized S-T, replace $SU(2)$ by max cpt subgp of some $\mathbb{C}$ Lie gp $\widehat{G}$. (Here G depends on problem.)

Entering last part

1. Families
   - Families in general
   - Distribution problems
   - Why interesting?
2. Equidistribution
   - General setup
   - Original Sato-Tate conjecture (for elliptic curves)
3. L-functions and automorphic forms
   - Automorphic forms and representations
   - Automorphic L-functions and their local invariants
   - Level and weight aspects
4. Equidistribution for automorphic families
   - Questions and Results

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Interesting statistical problems on $\{\mathcal{F}_k\}$ or $\{\mathfrak{F}_k\}$ $\cdots$

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Our concern $\cdots$ $\{\mathcal{F}_k\}_{k\geq 1}$ = family of auto reps of $GL_n(\mathbb{A})$.

- (level aspect) level $N_k \to \infty$, wt κ_k fixed, or
- (wt aspect) wt $\kappa_k \to \infty$, level N_k fixed.
- Have $|\mathcal{F}_k| < \infty$, $\lim_{k\to\infty} |\mathcal{F}_k| = \infty$.
- $\{\mathcal{F}_k\}_{k\geq 1} \rightsquigarrow \{\mathfrak{F}_k\}_{k\geq 1}$ family of L-functions

Interesting statistical problems on $\{\mathcal{F}_k\}$ or $\{\mathfrak{F}_k\}$ $\cdots$

- (equi-)dist of $(a_{p,1}(\pi), ..., a_{p,n}(\pi))$? (See next slide.)

Etc...

Statistical questions for automorphic families

$\{\mathcal{F}_k\}_{k\geq 1}$ = family of auto reps of $GL_n(\mathbb{A})$, level or wt aspect

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Question (Are the following μ-equidist for some μ on $(\mathbb{C}^\times)^n/S_n$?)

(Assume all $\pi \in \mathcal{F}_k$ are unramified at p or p_k.)

1. $\{t_p(\pi) : \pi \in \mathcal{F}_k\}_{k\geq 1}$, where p fixed,
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- Q2 (resp. Q3) is "S-T conj for families (resp. indiv aut reps)".
- Previous (beyond GL_2): Q1 (mainly cpt quot), Q2 (none).

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- Toy model: G fin gp $\rightsquigarrow$ $\mu^{\mathrm{pl}} = \sum_{\rho:\mathrm{irr\ rep}} (\dim \rho) \cdot \delta_\rho$ on fin set.

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Theorem (S.)

Let $\{\mathcal{F}_k\}$ be a family in level or wt aspect. If $G(\mathbb{R})$ admits a disc series (or an ell torus) then $\boxed{\lim_{k\to\infty} \mu^{\text{count}}_{\mathcal{F}_k,p} = \mu^{\text{pl}}_p}$.

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- Cor: Ramanujan conj holds at p for 100 percent of reps.

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Recall: $\mu^{\mathrm{count}}_{\mathcal{F}_k, p_k}$ captures the dist of p_k-compos of $\pi \in \mathcal{F}_k$.

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Our main theorem is:

Theorem (S.-Templier)

- *If $G(\mathbb{R})$ admits a disc series and*
- *$p_k \to \infty$ "slowly" relative to the growth of level or wt, then*

$$\boxed{\lim_{k\to\infty} \mu^{\mathrm{count}}_{\mathcal{F}_k, p_k} = \mu^{\mathrm{ST}}}.$$

Answer to Q2: Sato-Tate for families - (2)

$$\boxed{\lim_{k\to\infty} \mu^{\mathrm{count}}_{\mathcal{F}_k,p_k} = \mu^{\mathrm{ST}}} \quad \text{(S-T for families)}$$

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If f_p is an elt of unr Hecke alg for $G(\mathbb{Q}_p)$ of "exponent$\leq \delta$",

$$\mu^{\text{count}}_{\mathcal{F},p}(f_p) - \mu^{\text{pl}}_p(f_p) = \begin{cases} O(p^{a\delta}N^{-b}), & \text{level aspect,} \\ O(p^{c\delta}\kappa^{-d}), & \text{wt aspect,} \end{cases} \tag{1}$$

where a, b, c, d and const in $O(\cdot)$ are indep of p, e and N (or κ).

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Indeed,

- $\mu^{\mathrm{pl}}_{p_k} \to \mu^{\mathrm{ST}}$ as $k \to \infty$ (standard).
- p_k grows "slowly" rel to N_k or $\kappa_k \Rightarrow O(\cdots) \to 0$ as $k \to \infty$.

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(In fact, (1) implies answer to Q1 by fixing p and $N \to \infty$ or $\kappa \to \infty$.) How do we prove (1)?

Idea of proof: Thm with error terms (wt aspect)

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- remaining terms (error): count γ, bound vol, Φ^G, and orb-int.

Application to low-lying zeros in families

Theorem (S.-Templier)

The previous result plus quite a bit of work confirms:

the prediction of Katz-Sarnak about low-lying zero stats for families of automorphic L-functions via random matrix theory

for families of level or weight aspect considered in our Sato-Tate type theorem.

Remark

Probably the first time shown for L-fcns of arbitrarily high degree.

Thank You!