Cohomology of Bianchi Groups

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Bianchi groups and Bianchi manifolds

Definition

A *Bianchi group* Γ is a congruence subgroup of

$$\mathrm{SL}_2(\mathcal{O}_F)$$

where $F = \mathbb{Q}(\sqrt{d})$ with $d < 0$, and $\mathcal{O}_F$ is its ring of integers.

Let $E_k(\mathbb{C}) = \mathrm{Sym}^k(\mathbb{C}^2) \equiv \mathbb{C}[x, y]_k$ be the Γ-module.

For $g \in \mathrm{SL}_2(\mathbb{C})$ and $p(x, y) \in E_k(\mathbb{C})$, the action is defined by

$$(g.p)(x, y) = p((x, y).g^t);$$

Set $E_{k,l} = E_k(\mathbb{C}) \otimes \overline{E_l(\mathbb{C})}$.

We are interested in the cohomology groups

$$H^1(\Gamma, E_{k,l}(\mathbb{C})).$$

Cohomology groups

Theorem (Franke, 1998)

$H^1(\Gamma, E_{k,l}(\mathbb{C})) = \{$*Automorphic forms attached to* $\mathrm{Res}_{F/\mathbb{Q}}\boldsymbol{SL}_2$ *of cohomological type* $\}$

A Bianchi group Γ, as a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$, acts properly discontinuously on the hyperbolic 3-space

$$\mathbb{H} := \mathbb{C} \times \mathbb{R}^+.$$

If Γ is torsion free, then the action is free.

The quotient $Y_\Gamma := \Gamma\backslash\mathbb{H}$ is a non-compact, finite volume hyperbolic 3-manifold. Note that Y_Γ has no complex structure.

Cuspidal Cohomology

Let X_Γ be the Borel-Serre compactification of the quotient $Y_\Gamma := \Gamma\backslash\mathbb{H}$ with boundary ∂X_Γ.

Let $\mathcal{E}_{k,l}$ be the local system on Y_Γ induced by the representation $E_{k,l}$.

Let $\tilde{\mathcal{E}}_{k,l}$ be its extension on X_Γ.

We have for all i:

$$H^i(\Gamma, E_{k,l}(\mathbb{C})) \cong H^i(Y_\Gamma, \mathcal{E}_{k,l}) \cong H^i(X_\Gamma, \tilde{\mathcal{E}}_{k,l})$$

Definition

$$H^i_{cusp}(\Gamma, E_{k,l}) := Ker[H^i(X_\Gamma, \tilde{\mathcal{E}}_{k,l}) \to H^i(\partial X_\Gamma, \tilde{\mathcal{E}}_{k,l})]$$

Theorem (Harder)

$$H^i(\Gamma, E_{k,l}) = H^i_{cusp}(\Gamma, E_{k,l}) \oplus H^i_{Eis}(\Gamma, E_{k,l})$$

What do we know?

- $H^i(\Gamma, E_{k,l}(\mathbb{C})) = 0$ for $i > 2$;
- $H^1_{cusp}(\Gamma, E_{k,l}(\mathbb{C})) \cong H^2_{cusp}(\Gamma, E_{k,l}(\mathbb{C}))$;

Theorem (Borel and Wallach, 1970's)

If $k \neq l$, we have

$$H^1_{cusp}(\Gamma, E_{k,l}(\mathbb{C})) = 0.$$

So, we are interested in

$$H^1_{cusp}(\Gamma, E_{k,k}(\mathbb{C}))$$

Trivial Bound:

$$\dim H^1(\Gamma, E_{k,k}(\mathbb{C})) \ll_\Gamma k^2$$

Parallel weight: Upper Bounds

Conjecture

$$\dim H^1_{cusp}(\Gamma, E_{k,k}(\mathbb{C})) \sim_\Gamma k$$

Theorem (Finis, Grünewald, Triao, 2008)

$$\dim H^1_{cusp}(\Gamma, E_{k,k}(\mathbb{C})) \ll_\Gamma k^2 / \log k$$

Theorem (Calegari, Emerton, 2009)

$$\dim H^1_{cusp}(\Gamma(\mathfrak{p}^n), E) \ll_E \begin{cases} p^{2n}, & \text{if } p \text{ is unramified} \\ p^{5n}, & \text{if } p \text{ inerts} \end{cases}$$

Theorem (Marshall, 2010)

There exists a $\delta > 0$ such that

$$\dim H^1_{cusp}(\Gamma, E_{k,k}(\mathbb{C})) \ll_\Gamma k^{2-\delta}$$

Lower bounds

Theorem (Finis, Grünewald, Trio, 2008)

$$\dim H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}_F), E_{k,k}(\mathbb{C})) \gg k$$

Theorem (Sengün, T., 2011)

Let p be a rational prime and $\Gamma(p^n)$ be the principal congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_F)$. Then,

$$\dim H^1_{cusp}(\Gamma(p^n), E_{k,k}(\mathbb{C})) \gg_\Gamma k \quad \text{as the weight } k \text{ varies;}$$

and

$$\dim H^1_{cusp}(\Gamma(p^n), E_{k,k}(\mathbb{C})) \gg_E p^{3n} \quad \text{as the level } p^n \text{ varies.}$$

The Lefschetz number

Let $F = \mathbb{Q}(\sqrt{-d})$, $\Gamma \subseteq \mathrm{SL}_2(\mathcal{O}_F)$ be a Bianchi group and E be a Γ-module induced by a complex representation of $\mathrm{SL}_2(\mathbb{C})$.

Let $\tau \in Gal(F/\mathbb{Q})$, $\tau \neq 1$, i.e. the complex conjugation.

Then, τ acts on $\mathrm{SL}_2(\mathcal{O}_F)$. Assume that Γ is τ-stable.

Then, τ acts on Γ and E in a compatible way. Therefore, τ acts on the cohomology spaces

$$H^i(\Gamma, E).$$

The *Lefschetz number* is defined by

$$L(\tau, \Gamma, E) = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}(\tau \mid H^i(\Gamma, E)).$$

Therefore, if we denote $\tau^i = \operatorname{Tr}(\tau \mid H^i(\Gamma, E))$, then

$$L(\tau, \Gamma, E) = \tau^0 - \tau^1 + \tau^2$$

How to calculate the Lefschetz number

Let $\Gamma \subseteq \mathrm{SL}_2(\mathcal{O}_F)$ be τ-stable and Y_Γ^τ be the set of fixed points of τ.

Theorem (Rohlfs-Schwermer, 1998)

Assume that Γ is torsion free. Then,

$$L(\tau, \Gamma, E) = \chi(Y_\Gamma^\tau, \mathcal{E})$$

where

$$\chi(Y_\Gamma^\tau, \mathcal{E}) := \sum_i (-1)^i \mathrm{dim} H^i(Y_\Gamma^\tau, \mathcal{E})$$

The point is

$$Y_\Gamma^\tau = \bigcup_{\gamma \in H^1(G(F/\mathbb{Q}), \Gamma)} F(\gamma)$$

where $F(\gamma)$ is a either modular curve or a point.

An example

Let $F = \mathbb{Q}(\sqrt{-11})$, $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ and $E_{k,k} = \mathbb{C}$.

Then, $Y_\Gamma^\tau = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{C}^+ \sqcup \{Point\}$.

So, $L(\tau, \Gamma, \mathbb{C}) = \chi(Y_\Gamma^\tau) = 2$.

One can calculate that

$$\tau^0 = 1 \quad \text{and} \quad \tau^1_{Eis} = -1 \quad \text{and } \tau^2_{Eis} = 0.$$

Thus, we get

$$\tau^1_{cusp} = 0$$

and this implies that

$$\dim H^1_{cusp}(\Gamma, \mathbb{C}) \geq \tau^1_{cusp} = 0.$$

In fact,

$$H^1_{cusp}(\Gamma, \mathbb{C}) = 0$$

First result

Theorem (Sengün, T., 2011)

Let p be an odd, rational, unramified prime such that (p, d)=1. Then, for $n > 0$ we have

$$L(\tau, \Gamma(p^n), E_{k,k}) = \begin{cases} 2^t \cdot \dfrac{p^{3n+1} - p^{3n-1} - 12p^{2n+1} + 12p^{2n-1}}{12} \cdot (k+1) & \text{if } d \equiv 1(4) \\ 2^{t+1} \cdot \dfrac{p^{3n+1} - p^{3n-1} - 12p^{2n+1} + 12p^{2n-1}}{12} \cdot (k+1) & \text{else.} \end{cases}$$

where t is the number of distinct prime divisors of Disc($F/\mathbb{Q}$).

Second result

Theorem (Sengün, T., 2011)

Let $\mathfrak{a}$ *be a* τ*-stable ideal. Assume that* $N = N(\mathfrak{a} \cap \mathbb{Z}) > 2$*. Then, we have*

$$L(\tau, \Gamma(\mathfrak{a}), E_{k,k}) = (A + 2B)\frac{N^2(N-12)}{12}\prod_{p|N}(1-p^{-2}) \cdot (k+1)$$

where

d	j_2	A	B
$d \equiv 1(4)$	≥ 0	2^{t-s}	0
$d \equiv 2(4)$	0	2^{t-s}	2^{t-s-1}
	1	2^{t-s}	2^{t-s-1}
	2	$8 \cdot 2^{t-s}$	0
	≥ 3	$8 \cdot 2^{t-s-1}$	0
$d \equiv 3(4)$	0	2^{t-s}	2^{t-s-1}
	1	2^{t-s}	0
	2	$8 \cdot 2^{t-s}$	0
	$j_2 = 2n+1 \geq 3$	2^{t-s-1}	0
	$j_2 = 2n \geq 4$	$8 \cdot 2^{t-s-1}$	0