Modularity of Galois representations over imaginary quadratic fields

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- $\mathcal{O}$= ring of integers in a finite extension of $\mathbf{Q}_p$, ϖ=uniformizer, $\mathbf{F} = \mathcal{O}/\varpi$;
- Ψ= (unramified) Hecke character of ∞-type $\frac{z}{\overline{z}}$, $\Psi_\mathfrak{p} : G_\Sigma \to \mathcal{O}^\times$ the associated Galois character, $\chi_0 = \Psi_\mathfrak{p} \pmod{\varpi}$

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Then ρ *is modular, i.e.,*

$$L(\rho \otimes \gamma, s) = L(s, \pi)$$

for some automorphic representation π *of* $\mathrm{GL}_2(\mathbf{A}_F)$*.*

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$$\rho = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$$

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No such deformation ρ exists for F.

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- The condition $\dim_{\mathbf{F}} \mathrm{Ext}^1_{\mathbf{F}[G_\Sigma]}(\chi_0, \mathbf{1}) = 1$ is (probably) essential (work in progress).
- The unramifiedness of Ψ-condition can be replaced by demanding that $H^2_c(S_{K_f}, \mathbf{Z}_p)^{\mathrm{tors}} = 0$.

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- There exists a universal couple $(R_\Sigma, \rho_\Sigma : G_\Sigma \to \mathrm{GL}_2(R_\Sigma))$
- One gets a surjection $\phi : R_\Sigma \twoheadrightarrow \mathbf{T}_\Sigma$
- Goal: Show that ϕ is an isomorphism.

Ideal of reducibility

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$R_\Sigma / I_{\rm re}$ controls the *reducible* deformations

Key idea: Reduce the problem to that of modularity of reducible lifts.

Commutative algebra criterion

Theorem (Berger-K.)

Let R, S be commutative rings. Choose $r \in R$ such that $\bigcap_n r^n R = 0$. Let A be a domain and suppose that S is a finitely generated free module over A. Suppose we have a commutative diagrams of ring maps:

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow & & \downarrow \\ R/rR & \xrightarrow[\sim]{\overline{\phi}} & S/\phi(r)S. \end{array}$$

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The rank condition can be replaced by a condition $\frac{rR}{r^2R} \simeq \frac{\phi(r)S}{\phi(r)^2S}$ and then the theorem gives an alternative to the criterion of Wiles and Lenstra.

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Remark: The criterion removes the condition $p||B_{2,\omega^{k-2}}$ from a modularity result for residually reducible Galois representations over $\mathbf{Q}$ due to Calegari.

Principality of the ideal of reducibility

Theorem (Bellaïche-Chenevier, Calegari)

If

$$\dim_{\mathbf{F}} \mathrm{Ext}^1_{\mathbf{F}[G_\Sigma]}(\chi_0, \mathbf{1}) = \dim_{\mathbf{F}} \mathrm{Ext}^1_{\mathbf{F}[G_\Sigma]}(\mathbf{1}, \chi_0) = 1,$$

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Let A be a Noetherian local ring with $2 \in A^\times$. Set $S = A[G_\Sigma]$. Let $\rho : S \to M_2(A)$ be an A-algebra map with $\rho = \rho_0$ mod $\mathfrak{m}_A$.

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Higher-dimensional context

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  - the Bloch-Kato conjecture for the module $\mathrm{Hom}(\tilde{\tau}_2, \tilde{\tau}_1)$, where $\tilde{\tau}_j$ are the unique lifts of τ_j to $\mathcal{O}$,
  - Congruences among automorphic forms on higher-rank groups (results of Agarwal, Böcherer, Dummigan, Schulze-Pillot and K. on congruences to the Yoshida lifts on Sp_4 allow us to prove that certain 4-dimensional Galois representations arise from Siegel modular forms.)

Another modularity result

N=square-free, k=even, $p > k \geq 4$, $F = \mathbf{Q}$. Assume that every prime $l \mid N$ satisfies $l \not\equiv 1 \bmod p$. Let $f \in S_2(N), g \in S_k(N)$, $\Sigma = \{l \mid N, p\}$. Assume that $\overline{\rho}_f$ and $\overline{\rho}_g$ are absolutely irreducible.

Theorem (Berger-K.)

Suppose:

- $\dim_{\mathbf{F}} H^1_\Sigma(\mathbf{Q}, \mathrm{Hom}(\overline{\rho}_g, \overline{\rho}_f(k/2-1))) = 1$;
- $R_{\overline{\rho}_f(k/2-1)} = R_{\overline{\rho}_g} = \mathcal{O}$;
- *the B-K conjecture holds for* $H^1_\Sigma(\mathbf{Q}, \mathrm{Hom}(\rho_g, \rho_f(k/2-1)))$.

Let $\rho : G_{\mathbf{Q},\Sigma} \to \mathrm{GL}_4(\overline{\mathbf{Q}}_p)$ *be continuous, irreducible and such that:*

- $\overline{\rho}^{\mathrm{ss}} \cong \overline{\rho}_f(k/2-1) \oplus \overline{\rho}_g$;
- ρ *is crystalline at* p *and essentially self-dual.*

Then ρ *is modular. More precisely, there exists a Siegel modular form of weight* $k/2+1$*, level* $\Gamma_0(N)$ *and trivial character such that* $\rho \cong \rho_F$.

Thank you.

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