Constructions of Automorphic Forms

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December, 2011

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- For a group G, which is not GL_2, it is a big problem to construct Maass cusp forms.
- In this talk, we discuss recent progress on how to construct cuspidal automorphic representations of classical groups by means of residues of Eisenstein series and the relations with Langlands functoriality and the theory of endoscopy.

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- In this talk, I will discuss the general construction of endoscopy transfers and their relations with Arthur packets. This is my work in progress with Ginzburg and Soudry.

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- $L^2(X_G)$ denotes the space of functions: $\phi : X_G \to \mathbb{C}$ such that

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- $L^2(X_G)$ is a $G(\mathbb{A})$-module by $g \cdot \phi(x) := \phi(xg)$.

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$$L^2_d(X_G) = \oplus_{\pi \in \widehat{G(\mathbb{A})}} m_d(\pi) \cdot V_\pi$$

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- The **question** is to determine $m_d(\pi)$ explicitly.

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- For classical groups, the **Arthur Conjecture** asserts that

$$m_d(\pi) \leq \begin{cases} 1, & \text{if } G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n} \\ 2, & \text{if } G = \mathrm{SO}_{2n}. \end{cases}$$

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- $G = \mathrm{GL}_n$, $m_d(\pi) \leq 1$ (Shalika; Piatetski-Shapiro; Moeglin-Waldspurger);
- $G = \mathrm{SL}_2$, $m_d(\pi) \leq 1$ (Langlands-Lebasse; Ramkrishnan);
- $G = \mathrm{SL}_n (n \geq 3)$, $m_d(\pi) > 1$ for some π (Blasius; Lapid);
- $G = U_3$, $m_d(\pi) \leq 1$ (Rogawski);
- $G = G_2$, $m_d(\pi)$ unbounded for some family of π's (Gan-Gurevich-J.) and also (Gan);
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- $G = \mathrm{GSp}_4$, $m_d(\pi) \leq 1$ with π generic (J.-Soudry);
- The **Arthur Conjecture** is expected to be proved soon.

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- However, if ϕ and φ need to satisfy a particular relation, say, Langlands functoriality, for instance, it is in general a very hard problem to design the kernel function $\Theta(g,h)$!

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where $c(\pi_v)$ is a conjugacy class of semi-simple elements in the Langlands dual group ${}^LG$.
- ${}^LG = G^\vee(\mathbb{C}) \rtimes \Gamma_{\mathbb{Q}}$, where $G^\vee(\mathbb{C})$ is given by

$$\begin{array}{ccc} G & \Longleftrightarrow & (X, \Delta; X^\vee, \Delta^\vee) \\ \updownarrow & & \updownarrow \\ G^\vee(\mathbb{C}) & \Longleftrightarrow & (X^\vee, \Delta^\vee; X, \Delta) \end{array}$$

- $\mathrm{GL}_n^\vee(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SO}_{2n+1}^\vee(\mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{C})$.

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- **Problems:**
- (1) The image $c(\Pi^a(G))$ in $\mathcal{C}(G)$ (**Ramanujan Conjecture**).
- (2) The fiber $\Pi_{c(\pi)}$ (**refined structures of global packets**).

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- **Transfer:** G, H reductive algebraic $\mathbb{Q}$-groups and a group homomorphism

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which is compatible with the action of $\Gamma_{\mathbb{Q}}$. For any $\sigma \in \Pi^a(H)$, $\exists$ a $\pi \in \Pi^a(G)$ s.t.

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- The thickness of σ is defined in terms of invariant theory of ${}^LH$ and analytic properties of automorphic L-functions attached to σ, and was first introduced by Langlands in his Shaw prize lecture (2007, Shahidi's volume 2011).

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- This leads an explicit construction of the classical Shimura correspondence, which was the starting point of the classical theory of modular forms of half-integral weight.

Example: Theta Correspondence–Rallis Tower

- The theta correspondence may be formulated as

$$
\begin{array}{ccccc}
 & & \vdots & \vdots \\
 & & \mathrm{Sp}_6 & \Pi^a(\mathrm{Sp}_6) \\
 & \nearrow & \uparrow & \\
\Pi^a(\mathrm{SO}_{2m}) \quad \mathrm{SO}_{2m} & \longrightarrow & \mathrm{Sp}_4 & \Pi^a(\mathrm{Sp}_4) \\
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- **Questions:** What is the structure of the first occurrence?

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- C. Moeglin, in 2011, discussed the relation of theta correspondence, Adams's Conjecture, and Arthur's Conjecture on the discrete spectrum of automorphic forms.

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- Moeglin (1997); J.-Soudry (2007) proved the irreducibility of the first occurrence.

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- **Interesting applications:** (1) Counter-examples of the generalized Ramanujan conjecture; (2) Arithmeticity of special values of L-functions; (3) Nonvanishing of cohomology groups of certain degree over Shimura varieties; (4) Kudla's program on special cycles and generalized Gross-Zagier formula.

More Examples: Extended Theta Correspondences

- Replace the theta function $\Theta_{\Phi}^{\psi}(x)$ by the automorphic function Θ of reductive group $\mathbb{G}(\mathbb{A})$ attached to the minimal unipotant orbit of G and consider the extended Theta correspondence:

$$\int_{[G]\times[H]} \Theta(g,h)\phi(g)\varphi(h)dgdh \tag{3}$$

where $\phi \in \mathcal{A}(G)$ and $\varphi \in \mathcal{A}(H)$, and (G,H) forms a commuting pair in $\mathbb{G}$.

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- Important applications were obtained, including the work of Gross and Savin on the existence of motives whose Galois group is the exceptional group of type G_2.

More Examples: Partial Descents of Ginzburg-J.-Soudry

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- The transfers from $\mathrm{GL}_2 \times \mathrm{GL}_n$ to GL_{2n} is known for $n = 2$ (Ramakrishnan, 2000) and $n = 3$ (Kim-Shahidi, 2002).

Automorphic Descents of Ginzburg-Rallis-Soudry

- **Theorem of Cogdell, Kim, Piatetski-Shapiro and Shahidi:** Every irreducible generic cuspidal automorphic representation π of SO_{2n+1} has a Langlands functorial transfer to GL_{2n}, whose image is an irreducible automorphic representation τ.

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- The above work for all classical groups (CKPSS(2004) and the book of Ginzburg-Rallis-Soudry (2011)).

Discrete Spectrum: classical groups

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- **Theorem of Arthur:** Let G be a symplectic or orthogonal group. Then

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with the multiplicity $m_d(\pi)$ is 1 or 2.

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- $\Psi_2(G)$ is the set of global Arthur parameters of G.
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- For unitary groups and exceptional groups, it still needs to be worked out.

Arthur's Theorem and Langlands Functoriality

With the Langlands theory of Eisenstein series, the Arthur's theorem has following consequence to the **Langlands Functorial Transfer Conjecture** for classical groups G.

$$\Psi_2(G)$$

$$\psi$$

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- $\psi \mapsto \Pi(\psi)$ is given by Arthur stable trace formula.
- $\psi \mapsto E(\psi)$ is given by Langlands theory of Eisenstein series.
- $\Pi(\psi) \mapsto E(\psi)$ gives the existence of the Langlands transfer.

Two Problems Remain

Problems:

- (A) Refine the weak transfer $\Pi(\psi) \mapsto E(\psi)$ from classical group G to general linear group GL to the Langlands functorial transfer at all local places.

Possible Approach:

- (A) This is a deep arithmetic problem. At least, one needs the full theory of certain local L-functions and γ-factors. Also there is a serious problem with $E(\psi)$.

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Problems:

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- (B) Construct explicitly members in the Arthur packet $\Pi(\psi)$.

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- (A) This is a deep arithmetic problem. At least, one needs the full theory of certain local L-functions and γ-factors. Also there is a serious problem with $E(\psi)$.
- (B) We discuss recent progress of Ginzburg-J.-Soudry on construction of members in $\Pi(\psi)$, which is a generalization and combination of the theta liftings and the automorphic descents introduced by Ginzburg-Rallis-Soudry in 1998.

Global Arthur Parameters: $\Psi_2(G)$

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$$\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r$$

where $\psi_i = (\tau_i, b_i)$, with $\tau_i \in \Pi^{u,c,a}(\mathrm{GL}_{a_i})$ and $a_i, b_i \geq 1$.

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- For other classical groups, the description of ψ is similar.

Simple Arthur Parameter $\psi = (\tau, b)$

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Simple $\psi = (\tau, b)$-Tower with τ symplectic

$$
\begin{array}{ccccc}
\vdots & \uparrow & & \mathrm{SO}_{4nm} & (\tau, 2m) \\
 & & \swarrow & & \\
(\tau, 2m-1) & \mathrm{SO}_{4nm-2n+1} & & \uparrow & \\
 & & \searrow & & \\
 & \uparrow & & \mathrm{SO}_{4nm-4n} & (\tau, 2m-2) \\
 & & \swarrow & & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow & & \mathrm{SO}_{8n} & (\tau, 4) \\
 & & \swarrow & & \\
(\tau, 3) & \mathrm{SO}_{6n+1} & & \uparrow & \\
 & & \searrow & & \\
 & \uparrow & & \mathrm{SO}_{4n} & (\tau, 2) \\
 & & \swarrow & & \\
(\tau, 1) & \mathrm{SO}_{2n+1} & & &
\end{array}
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Simple $\psi = (\tau, b)$-Tower with τ symplectic, $L(\frac{1}{2}, \tau) \neq 0$

$$
\begin{array}{ccccc}
\vdots & \uparrow & & \mathrm{Sp}_{4nm} & (\tau, 2m) \\
 & & \swarrow & & \\
(\tau, 2m-1) & \widetilde{\mathrm{Sp}}_{4nm-2n} & & \uparrow & \\
 & & \searrow & & \\
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The first floor of the $\psi = (\tau, b)$-Tower is

$$\begin{array}{ccccc}
\mathrm{SO}_{4n} & & \overset{\mathrm{tc}}{\longleftrightarrow} & & \mathrm{Sp}_{4n} \\
 & \nwarrow \mathrm{lq} & & \mathrm{lq} \nearrow & \\
\mathrm{gg} \downarrow & & \mathrm{GL}_{2n} & & \downarrow \mathrm{fj} \\
 & \nearrow \mathrm{lt} & & \mathrm{lt} \nwarrow & \\
\mathrm{SO}_{2n+1} & & \overset{\mathrm{tc}}{\longleftrightarrow} & & \widetilde{\mathrm{Sp}}_{2n}
\end{array}$$

For the p-adic case, see Ginzburg-Rallis-Soudry (1999), J.-Soudry (2003), and J.-Nien-Qin (2010).

$\psi = (\tau, b)$-Tower with τ symplectic, $L(\frac{1}{2}, \tau) \neq 0$

The $m = (2l-1)$-th floor of the $\psi = (\tau, b)$-Theta Tower is

$$\begin{array}{ccccc}
\mathrm{SO}_{2nm+2n} & & \overset{\mathrm{tc}}{\longleftrightarrow} & & \mathrm{Sp}_{2nm+2n} \\
 & \nwarrow \mathrm{lq} & & \mathrm{lq} \nearrow & \\
\mathrm{gg} \downarrow & & \mathrm{GL}_{2nm} & & \downarrow \mathrm{fj} \\
 & \nearrow \mathrm{lt} & & \mathrm{lt} \nwarrow & \\
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\end{array}$$

Basic Triangles in a $\psi = (\tau, b)$-Tower

Consider the simple parameter $\psi = (\tau, b)$ with τ symplectic and $L(\frac{1}{2}, \tau) \neq 0$, and the following two **basic triangles**:

$$
\begin{array}{ccccc}
 & & & \mathrm{Sp}_{4nm+4n} & (\tau, 2m+2) \\
 & & \swarrow & & \\
(\tau, 2m+1) & \widetilde{\mathrm{Sp}}_{4nm+2n} & & \uparrow & \\
 & & \searrow & & \\
 & & & \mathrm{Sp}_{4nm} & (\tau, 2m)
\end{array}
$$

and

$$
\begin{array}{ccccc}
(\tau, 2m+1) & \widetilde{\mathrm{Sp}}_{4nm+2n} & & & \\
 & & \searrow & & \\
 & \uparrow & & \mathrm{Sp}_{4nm} & (\tau, 2m) \\
 & & \swarrow & & \\
(\tau, 2m-1) & \widetilde{\mathrm{Sp}}_{4nm-2n} & & &
\end{array}
$$

Basic Triangles in a $\psi = (\tau, b)$-Tower

- Define $G_b(\mathbb{A})$ by

$$G_b(\mathbb{A}) = \begin{cases} \widetilde{\mathrm{Sp}}_{4nm+2n}(\mathbb{A}) & \text{if } b = 2m+1; \\ \mathrm{Sp}_{4nm}(\mathbb{A}) & \text{if } b = 2m. \end{cases}$$

Then $G_b(\mathbb{A})$ has a standard parabolic subgroup

$$P_{1^n} = (\mathrm{GL}_1^{\times n} \times G_{b-1})U_{1^n}.$$

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  Then $G_b(\mathbb{A})$ has a standard parabolic subgroup

$$P_{1^n} = (\mathrm{GL}_1^{\times n} \times G_{b-1})U_{1^n}.$$

- Consider the Fourier-Jacobi coefficient along the unipotent radical of P_{1^n}: φ_b automorphic form on $G_b(\mathbb{A})$,

$$\mathrm{FJ}_{b-1}^{b}(\varphi_b, \psi)(h) := \int_{U_{1^n}(\mathbb{Q})\backslash U_{1^n}(\mathbb{A})} \varphi_b(uh)\widetilde{\theta}^{\psi}(uh)\psi_{U_{1^n}}(u)du.$$

  It is an automorphic form on $G_{b-1}(\mathbb{A})$.

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- Denote by $\mathcal{D}^b_{b-1,\psi}(\pi_b)$ the space generated by all Fourier-Jacobi coefficients $\mathrm{FJ}^b_{b-1}(\varphi_b, \psi)$ with all $\varphi_b \in \pi_b$.

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- We obtain

$$
\begin{array}{ccc}
 & & G_{b+2}(\mathbb{A}) \\
 & \mathcal{D}^{b+2}_{b+1,\psi} \swarrow & \\
G_{b+1}(\mathbb{A}) & & \uparrow \text{Res} \\
 & \mathcal{D}^{b+1}_{b,\psi^{-1}} \searrow & \\
 & & G_b(\mathbb{A})
\end{array}
$$

Basic Triangles in a $\psi = (\tau, b)$-Tower

- Take $\pi_b \in \Pi^a(G_b)$.
- Denote by $\mathcal{D}^b_{b-1,\psi}(\pi_b)$ the space generated by all Fourier-Jacobi coefficients $\mathrm{FJ}^b_{b-1}(\varphi_b, \psi)$ with all $\varphi_b \in \pi_b$.
- We obtain

$$
\begin{array}{ccc}
 & & G_{b+2}(\mathbb{A}) \\
 & \mathcal{D}^{b+2}_{b+1,\psi} \swarrow & \\
G_{b+1}(\mathbb{A}) & & \uparrow \mathrm{Res} \\
 & \mathcal{D}^{b+1}_{b,\psi^{-1}} \searrow & \\
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- Res is to take certain residue of Eisenstein series attached to the datum $\tau|\cdot|^s \otimes \pi_b$.

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- Res is to take certain residue of Eisenstein series attached to the datum $\tau|\cdot|^s \otimes \pi_b$.
- In general, this triangle is **NOT** commutative, and has no meaning related to Functoriality.

Basic Triangles in a $\psi = (\tau, b)$-Tower

Theorem (Ginzburg-J.-Soudry(2011))

Let π_b be the residual representation of $G_b(\mathbb{A})$ with Arthur parameter (τ, b), τ symplectic and $L(\frac{1}{2}, \tau) \neq 0$. Then $\mathrm{Res}(\pi_b)$ is the residual representation of $G_{b+2}(\mathbb{A})$ with Arthur parameter $(\tau, b+2)$, and $\mathcal{D}^{b+2}_{b+1,\psi}(\mathrm{Res}(\pi_b))$ is the residual representation of $G_{b+1}(\mathbb{A})$ with Arthur parameter $(\tau, b+1)$. Moreover, the basic triangle is a commutative diagram:

$$G_{b+2}(\mathbb{A}) \quad (\tau, b+2)$$

$$\mathcal{D}^{b+2}_{b+1,\psi} \swarrow$$

$$(\tau, b+1) \quad G_{b+1}(\mathbb{A}) \qquad \qquad \uparrow \textit{Res}$$

$$\mathcal{D}^{b+1}_{b,\psi^{-1}} \searrow$$

$$G_b(\mathbb{A}) \qquad (\tau, b)$$

Basic Triangles in a $\psi = (\tau, b)$-Tower

- When $b = 1$, we have the following triangle

$$
\begin{array}{ccccccc}
 & & (\tau,3) & \widetilde{\mathrm{Sp}}_{6n} & & & & \\
 & & & & \searrow & & & \\
 & & & \uparrow & & \mathrm{Sp}_{4n} & (\tau,2) & \mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi) \\
 & & & & \swarrow & & & \\
\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi) & & (\tau,1) & \widetilde{\mathrm{Sp}}_{2n} & & & &
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- $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ is the set of all irreducible, genuine, cuspidal automorphic representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which have τ as the ψ-weak Langlands transfer to $\mathrm{GL}_{2n}(\mathbb{A})$.

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- $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$ is the set of all irreducible automorphic representations π of $\mathrm{Sp}_{4n}(\mathbb{A})$, which occur in the discrete spectrum of $\mathrm{Sp}_{4n}(\mathbb{A})$, have the Arthur parameter $(\tau, 2)$, and with nonzero Fourier-Jacobi $\mathrm{FJ}_1^2(\varphi_\pi, \psi)$.

Basic Triangles in a $\psi = (\tau, b)$-Tower

- **Theorem of Ginzburg-J.-Soudry (2011):** Put $\Phi := \mathcal{D}^3_{2,\psi^{-1}} \circ \mathrm{Res}$ and $\Psi := \mathcal{D}^2_{1,\psi}$. Under a mild assumption, Φ and Ψ are bijections between $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ and $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$.

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- It is the first result which gives one-to-one relation between the set $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ of tempered cuspidal automorphic representations (assuming the generalized Ramanujan conjecture) and the set $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$ of non-tempered automorphic representations.

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- This is extension and refinement of the pioneer work of Piatetski-Shapiro, of Maass-Zagier, and of Andrianov on the Saito-Kurokawa conjecture (See also Ikeda 2006).

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- This is extension and refinement of the pioneer work of Piatetski-Shapiro, of Maass-Zagier, and of Andrianov on the Saito-Kurokawa conjecture (See also Ikeda 2006).
- This, combining with a work of Ginzburg-Rallis-Soudry (2005), proves the generalization of Duke-Imamoglu-Ikeda lifting, which constructs a non-tempered cuspidal automorphic forms of Sp_{2m} in terms of that of GL_2.

Basic Triangles in a $\psi = (\tau, b)$-Tower

- The proof of the theorem uses the commutativity of the whole diagram:

$$
\begin{array}{ccccc}
 & & & \mathrm{Sp}_{8n} & (\tau, 4) \\
 & & \swarrow & & \\
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 & & \searrow & & \\
 & \uparrow & & \mathrm{Sp}_{4n} & (\tau, 2) \\
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- It is highly nontrivial to show that for $\varphi \in \mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$, the following

$$\mathrm{Res} \circ \mathcal{D}^2_{1,\psi}(\varphi) = \mathcal{D}^4_{3,\psi} \circ \mathrm{Res}(\varphi)$$

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  hold as residual automorphic forms for some choice of data.
- The idea and the method are expected to work for $\psi = (\tau, b)$-towers of other classical groups.

General Constructions of Ginzburg-J.-Soudry

- Write a global Arthur parameter $\psi = (\tau, b) \boxplus \psi'$ with
  - $(\tau, b) \in \Psi_2(\mathrm{SO}_{2kb+1})$ ($\tau \in \mathcal{A}^{u,c}(\mathrm{GL}_{2k})$, $b = 2m + 1$);
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- For $\phi \in \mathcal{A}(\mathrm{SO}_{2l+1}), \varphi \in \mathcal{A}(\mathrm{SO}_{2l+2kb+1})$, if

$$\int \theta_{\tau;k,b,l}(g,h)\phi(g)\overline{\varphi(h)}dgdh \neq 0 \qquad (6)$$

then ϕ and φ are endoscopically related in terms of τ.

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  then ϕ and φ are endoscopically related in terms of τ.
- $\theta_{\tau;k,b,l}$ is a Fourier coefficient $\mathcal{E}_\tau^{\psi_V}$ of a residue $\mathcal{E}_\tau$ of certain Eisenstein series on $\mathrm{SO}_{2k(2l+b+1)}$, and the Fourier coefficient has stabilizer isomorphic to $\mathrm{SO}_{2l+1} \times \mathrm{SO}_{2l+2kb+1}$.

General Constructions of Ginzburg-J.-Soudry

The above construction is given by the following diagram.

$$\begin{array}{ccccc}
 & & & \mathrm{SO}_{2k(2l+b+1)} & \\
 & & \nearrow & \downarrow & \\
 & \mathrm{GL}_{2k} & & \theta_{\tau;k,b,l} & \\
\swarrow & & & \downarrow & \\
\mathrm{SO}_{2kb+1} & \times & \mathrm{SO}_{2l+1} & \longleftrightarrow & \mathrm{SO}_{2l+2kb+1} \\
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- For general classical groups, such a construction can be formulated in a similar way.

(GL_{2k}, τ)-Tower (Symplectic Type)

When τ is symplectic, b is odd; when τ is orthogonal, b is even.

$$\begin{array}{ccccc}
 & & \vdots & \vdots \\
 & & \mathrm{SO}_{2l+1+2k(b+2)} & \Pi(\psi_{\mathrm{SO}_{2l+1}} \boxplus (\tau, b+2)) \\
 & \nearrow & \uparrow & \\
\Pi(\psi_{\mathrm{SO}_{2l+1}}) \quad \mathrm{SO}_{2l+1} & \longrightarrow & \mathrm{SO}_{2l+1+2kb} & \Pi(\psi_{\mathrm{SO}_{2l+1}} \boxplus (\tau, b)) \\
 & \searrow & \uparrow & \\
 & & \mathrm{SO}_{2l+1+2k(b-2)} & \Pi(\psi_{\mathrm{SO}_{2l+1}} \boxplus (\tau, b-2)) \\
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(GL_{2k}, τ)-Tower (Orthogonal Type)

When τ is symplectic, b is even; when τ is orthogonal, b is odd.

$$
\begin{array}{ccccc}
 & & & \vdots & \vdots \\
 & & & \mathrm{Sp}_{2l+2k(b+2)} & \Pi(\psi_{\mathrm{SO}_{2l}} \boxplus (\tau, b+2)) \\
 & & \nearrow & \uparrow & \\
\Pi(\psi_{\mathrm{SO}_{2l}}) & \mathrm{SO}_{2l} & \longrightarrow & \mathrm{Sp}_{2l+2kb} & \Pi(\psi_{\mathrm{SO}_{2l}} \boxplus (\tau, b)) \\
 & & \searrow & \uparrow & \\
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(GL_m, τ)-Towers

- The dual diagram can be formulated, just like the theta correspondences for reductive dual pairs, by using the metaplectic double cover $\widetilde{\mathrm{Sp}}_{2l}$.

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- The local theory extends the **Howe duality principle**. Some work has been done through the work on the local descent by J.-Soudry (2003), J.-Nien-Qin (2010), and J.-Soudry (2011).

Tempered Cases

A tempered Arthur parameter $\psi \in \Psi_2(G)$ has form

$$\psi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1) = (\tau_1, 1) \boxplus \psi_2.$$

Then there exists an endoscopy group $H_1 \times H_2$ of G, such that $\psi_1 = (\tau_1, 1) \in \Psi_2(H_1)$ and $\psi_2 \in \Psi_2(H_2)$.

Theorem (Ginzburg (2008) and GJS (in progress))

Let π_1 be a generic member in $\Pi(\psi_1)$, π_2 be a generic member in $\Pi(\psi_2)$, and π be a generic member in $\Pi(\psi)$.

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- *The integral operator determines a descent from π to a generic member in $\Pi(\psi_2)$ by means of π_1.*

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- The explicit formula arose from such constructions has applications to other important problems, such as periods of automorphic forms, special values of L-functions, and so on.
- It is very interesting to find such constructions for Langlands functorial transfers which are **NOT** of endoscopy type.