#### Constructions of Automorphic Forms

#### Dihua Jiang University of Minnesota

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- It is always easy to construct Eisenstein series, but it is not that easy to construct cusp forms.
- ▶ For a group *G*, which is not GL<sub>2</sub>, it is a big problem to construct Maass cusp forms.
- In this talk, we discuss recent progress on how to construct cuspidal automorphic representations of classical groups by means of residues of Eisenstein series and the relations with Langlands functoriality and the theory of endoscopy.

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- In 2008, Ginzburg extended the idea of the descent method to cover the endoscopy transfers for irreducible generic cuspidal automorphic representations. The complete proof is an on-going work of Ginzburg-J.-Soudry
- In this talk, I will discuss the general construction of endoscopy transfers and their relations with Arthur packets. This is my work in progress with Ginzburg and Soudry.

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► G(Q) is a discrete subgroup of G(A), where A is the ring of adeles of Q. The quotient X<sub>G</sub> := Z<sub>G</sub>(A)G(Q)\G(A) has finite volume, where Z<sub>G</sub> is the center of G.

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- ►  $L^2(X_G)$  denotes the space of functions:  $\phi$  :  $X_G \to \mathbb{C}$  such that

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•  $L^2(X_G)$  is a  $G(\mathbb{A})$ -module by  $g \cdot \phi(x) := \phi(xg)$ .

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$$L^2_d(X_G) = \oplus_{\pi \in \widehat{G(\mathbb{A})}} m_d(\pi) \cdot V_{\pi}$$

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• The **question** is to determine  $m_d(\pi)$  explicitly.

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For classical groups, the Arthur Conjecture asserts that

$$m_d(\pi) \leq \begin{cases} 1, & \text{if } G = SO_{2n+1}, Sp_{2n} \\ 2, & \text{if } G = SO_{2n}. \end{cases}$$

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- G = GL<sub>n</sub>, m<sub>d</sub>(π) ≤ 1 (Shalika; Piatetski-Shapiro; Moeglin-Waldspurger);
- $G = SL_2$ ,  $m_d(\pi) \le 1$  (Langlands-Lebasse; Ramkrishnan);
- $G = SL_n (n \ge 3)$ ,  $m_d(\pi) > 1$  for some  $\pi$  (Blasius; Lapid);
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- G = G<sub>2</sub>, m<sub>d</sub>(π) unbounded for some family of π's (Gan-Gurevich-J.) and also (Gan);
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- > The Arthur Conjecture is expected to be proved soon.

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- Let Θ(g, h) be an automorphic function on G(A) × H(A) and φ, φ be automorphic forms on G(A), H(A), respectively. Consider the following integral (assuming convergence)

$$\int_{[G]\times[H]} \Theta(g,h)\phi(g)\varphi(h)dgdh.$$
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- If (1) is nonzero, then the integration along *dh* will construct automorphic functions on *G*(A) by means of those on *H*(A), while the integration along *dg* will produces the opposite direction construction. Hence the construction is easy!
- However, if φ and φ need to satisfy a particular relation, say, Langlands functoriality, for instance, it is in general a very hard problem to design the kernel function Θ(g, h)!

 Automorphic representations are G(A)-submodules in L<sup>2</sup>(X<sub>G</sub>). The set of all irreducible ones is denoted by Π<sup>a</sup>(G).

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- Following Harish-Chandra and Bernstein, any π ∈ Π<sup>a</sup>(G) can be written as π ≅ ⊗<sub>ν</sub>π<sub>ν</sub>, where π<sub>ν</sub> is an irreducible admissible representation of G(Q<sub>ν</sub>).

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- By Satake theory of spherical functions, for almost all v,

$$\pi_{\mathbf{v}} \iff \mathbf{c}(\pi_{\mathbf{v}})$$

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•  ${}^{L}G = G^{\vee}(\mathbb{C}) \rtimes \Gamma_{\mathbb{Q}}$ , where  $G^{\vee}(\mathbb{C})$  is given by

- S denotes any finite set of primes p and  $\infty$ .
- ►  $c(S) := \{c_v \mid v \notin S\}$ , where  $c_v$  is a s.-s. conjugacy class <sup>L</sup>G.

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- Π<sup>a</sup>(G): the set of equivalence classes of irreducible automorphic representations of G(A).

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- $\pi = \otimes_{\nu} \pi_{\nu} \in \Pi^{a}(G) \Longrightarrow c(S_{\pi})$  for some finite set  $S_{\pi}$  of places.
- ▶  $\exists$  a map c :  $\pi \mapsto c(\pi)$  from  $\Pi^{a}(G)$  to  $\mathcal{C}(G)$ .
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- Problems:
- ► (1) The image  $c(\Pi^a(G))$  in C(G) (Ramanujan Conjecture).
- (2) The fiber  $\Pi_{c(\pi)}$  (refined structures of global packets).

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# Langlands Functoriality

Langlands Functoriality Conjecture consists of Two Parts:

► **Transfer:** *G*, *H* reductive algebraic Q-groups and a group homomorphism

$$\rho : {}^{L}H \to {}^{L}G,$$

which is compatible with the action of  $\Gamma_{\mathbb{Q}}$ . For any  $\sigma \in \Pi^{a}(H)$ ,  $\exists a \pi \in \Pi^{a}(G)$  s.t.

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Thickness: For each tempered π ∈ Π<sup>a</sup>(G), ∃ an H; a thick σ ∈ Π<sup>a</sup>(H), s.t. π is a Langlands functorial transfer of σ.

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- Thickness: For each tempered π ∈ Π<sup>a</sup>(G), ∃ an H; a thick σ ∈ Π<sup>a</sup>(H), s.t. π is a Langlands functorial transfer of σ.
- The thickness of σ is defined in terms of invariant theory of <sup>L</sup>H and analytic properties of automorphic L-functions attached to σ, and was first introduced by Langlands in his Shaw prize lecture (2007, Shahidi's volume 2011).

►  $O_{2m} \times \operatorname{Sp}_{2n} \to \operatorname{Sp}_{4mn}$ , via the tensor product, forms a reductive dual pair in  $\operatorname{Sp}_{4mn}$  in the sense of R. Howe.

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- The theta correspondence is given by

$$\int_{[O_{2m}]\times[\operatorname{Sp}_{2n}]} \Theta_{\Phi}^{\psi}(g,h)\phi(g)\varphi(h)dgdh \qquad (2)$$

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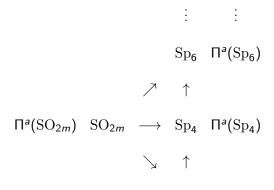
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where  $\phi \in \mathcal{A}(O_{2m})$  and  $\varphi \in \mathcal{A}(\mathrm{Sp}_{2n})$ .

This leads an explicit construction of the classical Shimura correspondence, which was the starting point of the classical theory of modular forms of half-integral weight.

#### Example: Theta Correspondence-Rallis Tower

The theta correspondence may be formulated as



 $Sp_2 \quad \Pi^a(Sp_2)$ 

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Questions: What is the structure of the first occurrence?

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- S. Kudla, in 1986, the local version of the Tower Properties.

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- J.-P. Waldspurger, 1980, the representation-theoretic approach to investigate the Shimura correpsondence.
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- C. Moeglin, in 2011, discussed the relation of theta correspondence, Adams's Conjecture, and Arthur's Conjecture on the discrete spectrum of automorphic forms.

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- Interesting applications: (1) Counter-examples of the generalized Ramanujan conjecture; (2) Arithmeticity of special values of *L*-functions; (3) Nonvanishing of cohomology groups of certain degree over Shimura varieties; (4) Kudla's program on special cycles and generalized Gross-Zagier formula.

## More Examples: Extended Theta Correspondences

► Replace the theta function Θ<sup>ψ</sup><sub>Φ</sub>(x) by the automorphic function Θ of reductive group G(A) attached to the minimal unipotant orbit of G and consider the extended Theta correspondence:

$$\int_{[G]\times[H]} \Theta(g,h)\phi(g)\varphi(h)dgdh$$
(3)

where  $\phi \in \mathcal{A}(G)$  and  $\varphi \in \mathcal{A}(H)$ , and (G, H) forms a commuting pair in  $\mathbb{G}$ .

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- Important applications were obtained, including the work of Gross and Savin on the existence of motives whose Galois group is the exceptional group of type G<sub>2</sub>.

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- ▶ The transfers from  $GL_2 \times GL_n$  to  $GL_{2n}$  is known for n = 2 (Ramakrishnan, 2000) and n = 3 (Kim-Shahidi, 2002).

### Automorphic Descents of Ginzburg-Rallis-Soudry

Theorem of Cogdell, Kim, Piatetski-Shapiro and Shahidi: Every irreducible generic cuspidal automorphic representation π of SO<sub>2n+1</sub> has a Langlands functorial transfer to GL<sub>2n</sub>, whose image is an irreducible automorphic representation τ.

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 The above work for all classical groups (CKPSS(2004) and the book of Ginzburg-Rallis-Soudry (2011)).

### Discrete Spectrum: classical groups

With the recent progress on the Fundamental Lemma and its variants by Ngo (and Waldspurger, Laumon-Ngo, ...), the stable trace formula of Arthur is able to prove the following key theorem, which was announced in Arthur's 2005 Clay lecture notes and forthcoming book 2011.

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- Theorem of Arthur: Let G be a symplectic or orthogonal group. Then

$$L^2_d(X_G) = \bigoplus_{\psi \in \Psi_2(G)} m_d(\psi) \left( \bigoplus_{\pi \in \Pi(\psi), m_d(\pi) \neq 0} V_{\pi} \right)$$

with the multiplicity  $m_d(\pi)$  is 1 or 2.

- $\Psi_2(G)$  is the set of global Arthur parameters of G.
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- For unitary groups and exceptional groups, it still needs to be worked out.

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### Arthur's Theorem and Langlands Functoriality

With the Langlands theory of Eisenstein series, the Arthur's theorem has following consequence to the Langlands Functorial Transfer Conjecture for classical groups *G*.

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•  $\psi \mapsto \Pi(\psi)$  is given by Arthur stable trace formula.

- $\psi \mapsto E(\psi)$  is given by Langlands theory of Eisenstein series.
- $\Pi(\psi) \mapsto E(\psi)$  gives the existence of the Langlands transfer.

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Problems:

(A) Refine the weak transfer Π(ψ) → E(ψ) from classical group G to general linear group GL to the Langlands functorial transfer at all local places.

Possible Approach:

 (A) This is a deep arithmetic problem. At least, one needs the full theory of certain local *L*-functions and γ-factors. Also there is a serious problem with *E*(ψ).

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Problems:

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  - (B) We discuss recent progress of Ginzburg-J.-Soudry on construction of members in Π(ψ), which is a generalization and combination of the theta liftings and the automorphic descents introduced by Ginzburg-Rallis-Soudry in 1998.

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• Take 
$$G = SO_{2n+1}$$
, then  $G^{\vee}(\mathbb{C}) = Sp_{2n}(\mathbb{C})$ .

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- ► Each ψ ∈ Ψ<sub>2</sub>(SO<sub>2n+1</sub>) (Arthur parameters) is written as a formal sum of stable Arthur parameters:

$$\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r$$

where  $\psi_i = (\tau_i, b_i)$ , with  $\tau_i \in \Pi^{u,c,a}(GL_{a_i})$  and  $a_i, b_i \ge 1$ .

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$$\begin{array}{rcl} \mathrm{SO}_{a_1 \cdot b_1 + 1} & \times & \cdots & \times & \mathrm{SO}_{a_r \cdot b_r + 1} \implies & \mathrm{SO}_{2n + 1} \\ \Pi(\psi_1) & \otimes & \cdots & \otimes & \Pi(\psi_r) \implies & \Pi(\psi) \end{array}$$

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• For other classical groups, the description of  $\psi$  is similar.

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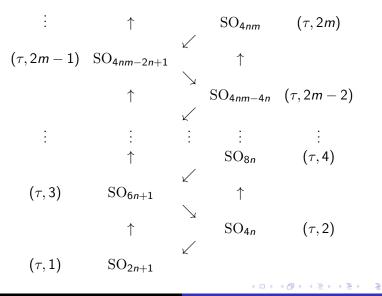
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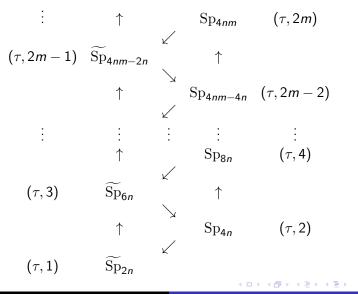
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#### Simple $\psi = (\tau, b)$ -Tower with $\tau$ symplectic



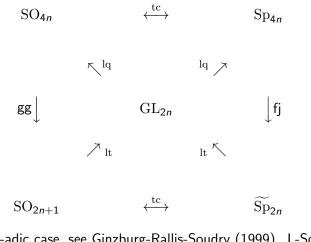
# Simple $\psi = (\tau, b)$ -Tower with $\tau$ symplectic, $L(\frac{1}{2}, \tau) \neq 0$



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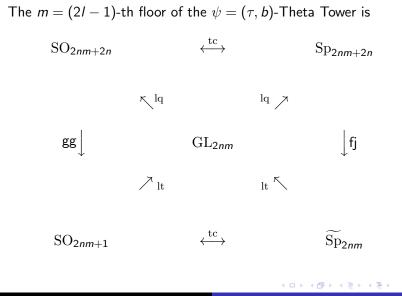
# $\psi = ( au, b)$ -Tower with au symplectic, $L(rac{1}{2}, au) eq 0$

The first floor of the  $\psi = (\tau, b)$ -Tower is



For the p-adic case, see Ginzburg-Rallis-Soudry (1999), J.-Soudry (2003), and J.-Nien-Qin (2010).

# $\psi = (\tau, b)$ -Tower with au symplectic, $L(\frac{1}{2}, \tau) \neq 0$



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Consider the simple parameter  $\psi = (\tau, b)$  with  $\tau$  symplectic and  $L(\frac{1}{2}, \tau) \neq 0$ , and the following two **basic triangles**:

$$\begin{array}{ccc} & & \operatorname{Sp}_{4nm+4n} & (\tau, 2m+2) \\ (\tau, 2m+1) & \widetilde{\operatorname{Sp}}_{4nm+2n} & & \uparrow \\ & & & & \\ & & &$$

and

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• Define  $G_b(\mathbb{A})$  by

$$G_b(\mathbb{A}) = \begin{cases} \widetilde{\operatorname{Sp}}_{4nm+2n}(\mathbb{A}) & \text{if } b = 2m+1; \\ \operatorname{Sp}_{4nm}(\mathbb{A}) & \text{if } b = 2m. \end{cases}$$

Then  $G_b(\mathbb{A})$  has a standard parabolic subgroup

$$P_{1^n} = (\mathrm{GL}_1^{\times n} \times G_{b-1}) U_{1^n}.$$

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$$P_{1^n} = (\mathrm{GL}_1^{\times n} \times G_{b-1}) U_{1^n}.$$

 Consider the Fourier-Jacobi coefficient along the unipotent radical of P<sub>1<sup>n</sup></sub>: φ<sub>b</sub> automorphic form on G<sub>b</sub>(A),

$$\mathrm{FJ}_{b-1}^{b}(\varphi_{b},\psi)(h) := \int_{\mathcal{U}_{1^{n}}(\mathbb{Q})\setminus\mathcal{U}_{1^{n}}(\mathbb{A})} \varphi_{b}(uh)\widetilde{ heta}^{\psi}(uh)\psi_{\mathcal{U}_{1^{n}}}(u)du.$$

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It is an automorphic form on  $G_{b-1}(\mathbb{A})$ .

• Take  $\pi_b \in \Pi^a(G_b)$ .

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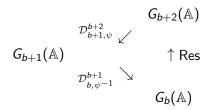
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▶ Denote by D<sup>b</sup><sub>b-1,ψ</sub>(π<sub>b</sub>) the space generated by all Fourier-Jacobi coefficients FJ<sup>b</sup><sub>b-1</sub>(φ<sub>b</sub>, ψ) with all φ<sub>b</sub> ∈ π<sub>b</sub>.

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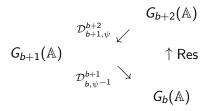
- ▶ Denote by D<sup>b</sup><sub>b-1,ψ</sub>(π<sub>b</sub>) the space generated by all Fourier-Jacobi coefficients FJ<sup>b</sup><sub>b-1</sub>(φ<sub>b</sub>, ψ) with all φ<sub>b</sub> ∈ π<sub>b</sub>.
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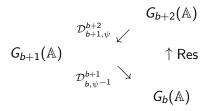


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- Res is to take certain residue of Eisenstein series attached to the datum τ | · |<sup>s</sup> ⊗ π<sub>b</sub>.
- In general, this triangle is NOT commutative, and has no meaning related to Functoriality.

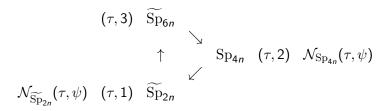
#### Theorem (Ginzburg-J.-Soudry(2011))

Let  $\pi_b$  be the residual representation of  $G_b(\mathbb{A})$  with Arthur parameter  $(\tau, b)$ ,  $\tau$  symplectic and  $L(\frac{1}{2}, \tau) \neq 0$ . Then  $\operatorname{Res}(\pi_b)$  is the residual representation of  $G_{b+2}(\mathbb{A})$  with Arthur parameter  $(\tau, b + 2)$ , and  $\mathcal{D}_{b+1,\psi}^{b+2}(\operatorname{Res}(\pi_b))$  is the residual representation of  $G_{b+1}(\mathbb{A})$  with Arthur parameter  $(\tau, b + 1)$ . Moreover, the basic triangle is a commutative diagram:

$$( au,b+1) \quad G_{b+1}(\mathbb{A}) \quad egin{array}{c} \mathcal{G}_{b+2}(\mathbb{A}) & ( au,b+2) \\ \mathcal{D}_{b+1,\psi}^{b+2} & \swarrow & & \\ \mathcal{D}_{b,\psi^{-1}}^{b+1} & \searrow & & \\ \mathcal{D}_{b,\psi^{-1}}^{b+1} & \searrow & & & \\ \mathcal{G}_{b}(\mathbb{A}) & ( au,b) \end{array}$$

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*N*<sub>Sp<sub>2n</sub></sub>(τ, ψ) is the set of all irreducible, genuine, cuspidal automorphic representations of Sp<sub>2n</sub>(A), which have τ as the ψ-weak Langlands transfer to GL<sub>2n</sub>(A).

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$$\begin{array}{cccc} (\tau,3) & \widetilde{\operatorname{Sp}}_{6n} & & & \\ & \uparrow & & & \\ & \uparrow & & & \\ & & \swarrow & & \\ \mathcal{N}_{\widetilde{\operatorname{Sp}}_{2n}}(\tau,\psi) & (\tau,1) & \widetilde{\operatorname{Sp}}_{2n} & & \end{array}$$

- $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$  is the set of all irreducible, genuine, cuspidal automorphic representations of  $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ , which have  $\tau$  as the  $\psi$ -weak Langlands transfer to  $\mathrm{GL}_{2n}(\mathbb{A})$ .
- ►  $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$  is the set of all irreducible automorphic representations  $\pi$  of  $\mathrm{Sp}_{4n}(\mathbb{A})$ , which occur in the discrete spectrum of  $\mathrm{Sp}_{4n}(\mathbb{A})$ , have the Arthur parameter  $(\tau, 2)$ , and with nonzero Fourier-Jacobi  $\mathrm{FJ}_{1}^{2}(\varphi_{\pi}, \psi)$ .

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#### Theorem of Ginzburg-J.-Soudry (2011): Put

$$\begin{split} \Phi &:= \mathcal{D}^3_{2,\psi^{-1}} \circ \operatorname{Res} \text{ and } \Psi := \mathcal{D}^2_{1,\psi}. \text{ Under a mild assumption,} \\ \Phi \text{ and } \Psi \text{ are bijections between } \mathcal{N}_{\widetilde{\operatorname{Sp}}_{2}}(\tau,\psi) \text{ and } \mathcal{N}_{\operatorname{Sp}_{4g}}(\tau,\psi). \end{split}$$

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- Theorem of Ginzburg-J.-Soudry (2011): Put  $\Phi := \mathcal{D}_{2,\psi^{-1}}^3 \circ \operatorname{Res}$  and  $\Psi := \mathcal{D}_{1,\psi}^2$ . Under a mild assumption,  $\Phi$  and  $\Psi$  are bijections between  $\mathcal{N}_{\widetilde{\operatorname{Sp}}_{2n}}(\tau,\psi)$  and  $\mathcal{N}_{\operatorname{Sp}_{4n}}(\tau,\psi)$ .
- ▶ It is the first result which gives one-to-one relation between the set  $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$  of tempered cuspidal automorphic representations (assuming the generalized Ramanujan conjecture) and the set  $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau,\psi)$  of non-tempered automorphic representations.

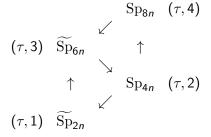
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- This is extension and refinement of the pioneer work of Piatetski-Shapiro, of Maass-Zagier, and of Andrianov on the Saito-Kurokawa conjecture (See also Ikeda 2006).

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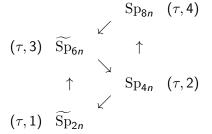
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- This is extension and refinement of the pioneer work of Piatetski-Shapiro, of Maass-Zagier, and of Andrianov on the Saito-Kurokawa conjecture (See also Ikeda 2006).
- This, combining with a work of Ginzburg-Rallis-Soudry (2005), proves the generalization of Duke-Imamoglu-Ikeda lifting, which constructs a non-tempered cuspidal automorphic forms of Sp<sub>2m</sub> in terms of that of GL<sub>2</sub>.

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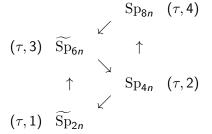
It is highly nontrivial to show that for φ ∈ N<sub>Sp4n</sub>(τ, ψ), the following

$$\operatorname{Res} \circ \mathcal{D}^2_{1,\psi}(\varphi) = \mathcal{D}^4_{3,\psi} \circ \operatorname{Res}(\varphi)$$

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The idea and the method are expected to work for ψ = (τ, b)-towers of other classical groups.

#### General Constructions of Ginzburg-J.-Soudry

- Write a global Arthur parameter  $\psi = (\tau, b) \boxplus \psi'$  with
  - $(\tau, b) \in \Psi_2(\mathrm{SO}_{2kb+1}) \ (\tau \in \mathcal{A}^{u,c}(\mathrm{GL}_{2k}), \ b = 2m+1);$
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- Look for an endoscopy structure

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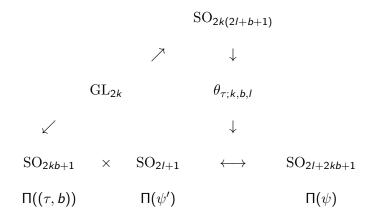
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►  $\theta_{\tau;k,b,l}$  is a Fourier coefficient  $\mathcal{E}_{\tau}^{\psi_{V}}$  of a residue  $\mathcal{E}_{\tau}$  of certain Eisenstein series on  $SO_{2k(2l+b+1)}$ , and the Fourier coefficient has stabilizer isomorphic to  $SO_{2l+1} \times SO_{2l+2kb+1}$ .

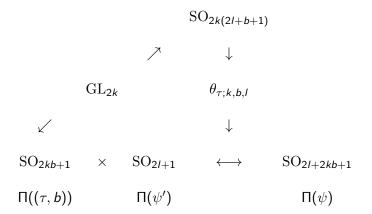
The above construction is given by the following diagram.



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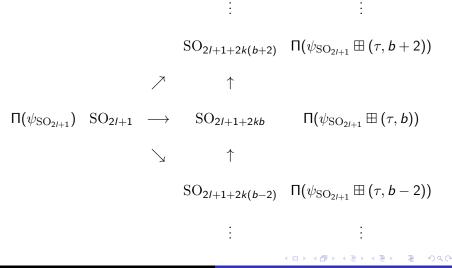
The above construction is given by the following diagram.



 For general classical groups, such a construction can be formulated in a similar way.

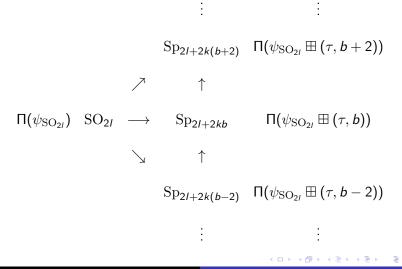
### $(GL_{2k}, \tau)$ -Tower (Symplectic Type)

When  $\tau$  is symplectic, b is odd; when  $\tau$  is orthogonal, b is even.



### $(GL_{2k}, \tau)$ -Tower (Orthogonal Type)

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► The dual diagram can be formulated, just like the theta correspondences for reductive dual pairs, by using the metaplectic double cover Sp<sub>21</sub>.

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- The local theory extends the Howe duality principle. Some work has been done through the work on the local descent by J.-Soudry (2003), J.-Nien-Qin (2010), and J.-Soudry (2011).

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A tempered Arthur parameter  $\psi \in \Psi_2(G)$  has form

$$\psi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1) = (\tau_1, 1) \boxplus \psi_2.$$

Then there exists an endoscopy group  $H_1 \times H_2$  of G, such that  $\psi_1 = (\tau_1, 1) \in \Psi_2(H_1)$  and  $\psi_2 \in \Psi_2(H_2)$ .

Theorem (Ginzburg (2008) and GJS (in progress))

Let  $\pi_1$  be a generic member in  $\Pi(\psi_1)$ ,  $\pi_2$  be a generic member in  $\Pi(\psi_2)$ , and  $\pi$  be a generic member in  $\Pi(\psi)$ .

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- The integral operator determines a descent from π to a generic member in Π(ψ<sub>2</sub>) by means of π<sub>1</sub>.

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- It is very interesting to find such constructions for Langlands functorial transfers which are **NOT** of endoscopy type.