# On Saito-Kurokawa Lifts

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Recall

$$\iota_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} : {}^t g \iota_n g = \mu_n(g)\iota_n, \quad \mu_n(g) \in \mathrm{GL}_1\}$$

and

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# $GSp_{2n}$ and The Siegel Upper Half Space

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Siegel upper half space of degree $n$ is given by

$$\mathfrak{h}^n = \{Z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t Z = Z, \mathrm{Im}(Z) > 0\}.$$

# Action of $\mathrm{GSp}_{2n}^{+}(\mathbb{R})$ on $\mathfrak{h}^n$

$\mathrm{GSp}_{2n}^{+}(\mathbb{R}) = \{\gamma \in \mathrm{GSp}_{2n}(\mathbb{R}) : \mu_n(\gamma) > 0\}$ action on $\mathfrak{h}^n$ is given by

$$\gamma Z = (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}$$

for $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \mathrm{GSp}_{2n}^{+}(\mathbb{R})$ and $Z \in \mathfrak{h}^n$.

# Congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$

For $M \geq 1$,

**Definition**

$$\Gamma_0^{(n)}(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}) : c \equiv 0 \pmod{M} \right\}.$$

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Note this is the natural generalization of $\Gamma_0(M) \subset \mathrm{SL}_2(\mathbb{Z})$ to this setting.

# Action of $\mathrm{GSp}_{2n}^+(\mathbb{R})$ on functions $F : \mathfrak{h}^n \to \mathbb{C}$

For $\gamma \in \mathrm{GSp}_{2n}^+(\mathbb{R})$ and $Z \in \mathfrak{h}^n$,

$$j(\gamma, Z) = \det(c_\gamma Z + d_\gamma).$$

Let $\kappa$ be a positive integer. Given a function $F : \mathfrak{h}^n \to \mathbb{C}$, we set

$$(F|_\kappa \gamma)(Z) = \mu_n(\gamma)^{n\kappa/2} j(\gamma, Z)^{-\kappa} F(\gamma Z).$$

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**Definition**

We say such an $F$ is a Siegel modular form of degree $n$, weight $\kappa$, and level $\Gamma$ if $F$ is a holomorphic function and satisfies

$$(F|_\kappa \gamma)(Z) = F(Z)$$

for all $\gamma \in \Gamma$.
The space of Siegel modular forms of weight $\kappa$ and level $\Gamma$ is $M_\kappa(\Gamma)$.

# Fourier Series of a Siegel Modular Form

If $F$ is a Siegel modular form of degree $n > 1$, it has a Fourier expansion of the form

$$F(Z) = \sum_{T \in \mathbb{S}_n^{\geq 0}(\mathbb{Z})} a_F(T) e(\mathrm{Tr}(TZ))$$

where $\mathbb{S}_n^{\geq 0}(\mathbb{Z})$ is the semi-group of $nxn$ positive semi-definite half-integral symmetric matrices.

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**Definition**

$F$ is a Siegel cusp form ($F \in S_\kappa(\Gamma)$ ) if and only if $a_F(T) = 0$ when $\det T = 0$ and

$$F(Z) = \sum_{T \in \mathbb{S}_n^{\geq 0}(\mathbb{Z}), T > 0} a_F(T) e(\mathrm{Tr}(TZ)).$$

# Paramodular Forms

Consider the Siegel upper half plane $\mathfrak{h}^2$ of degree 2 and the following form

**Definition**

Let $t \geq 1$ be an integer and let

$$P_t = \text{diag}(1, t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

and we consider the skew-symmetric bilinear form written in block form

$$J_t = \begin{pmatrix} 0 & P_t \\ -P_t & 0 \end{pmatrix}$$

# The Paramodular Group of level $t \geq 1$

## Definition

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$$\Gamma[t] = \mathrm{Sp}_4(\mathbb{Q}) \bigcap \left\{ \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$

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Note that for $t = 1$ this group is the Siegel Modular group $\mathrm{Sp}_4(\mathbb{Z})$.

# The Paramodular Group in a Geometric Context

The quotient space

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We may write $S$ as a two dimensional complex torus

$$\mathbb{C}^2/L$$

where

$$L = Z\mathbb{Z}^2 \oplus P_t\mathbb{Z}^2$$

$Z \in \mathfrak{h}^2$ and $P_t = \mathrm{diag}(1, t)$.

The polarization with respect to this basis is given by the form $J_t$.

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**Conjecture**

*There is a one-to-one correspondence between isogeny classes of abelian surfaces $A_{/\mathbb{Q}}$ of conductor $t$ with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and weight 2 and level $\Gamma[t]$ newforms $F$ with rational eigenvalues, not in the span of Gritsenko lifts, such that $L(s, A) = L(s, F, \mathrm{spin})$.*

# The Jacobi group

Let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.
The Jacobi group $\Gamma^J := \Gamma \ltimes \mathbb{Z}^2$ is

$$\Gamma^J := \{(M, X) : (M, X)(M', X') = (MM', XM' + X')\}$$

for all $M, M' \in \Gamma$ and $X = [\lambda, \mu], X' = [\alpha, \beta] \in \mathbb{Z}^2$.

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If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{Z})^J$ is called the the full Jacobi group.

The space of Jacobi cusp forms of weight $\kappa$, index $t$ and level $\Gamma_0(M)^J$ (resp $\mathrm{SL}_2(\mathbb{Z})^J$) is $J^c_{\kappa,t}(\Gamma_0(M)^J)$ (resp. $J^c_{\kappa,t}$).

# Classical Saito-Kurokawa lifts

Let $f \in S^{\mathrm{new}}_{2\kappa-2}(\Gamma_0(m))$ be a newform. There are essentially two classical ways to construct a Siegel modular form $F_f$ associated to $f$ (referred to as a Saito-Kurokawa lift of $f$):

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1. If $\kappa$ is even, one has a lifting $F_f \in S_\kappa(\Gamma_0^{(2)}(m))$ due to numerous people: Manickham-Ramakrishnan-Vasudevan, Piatetski-Shapiro, Schmidt, Skinner-Urban.

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2. If $f \in S_{2\kappa-2}^{\text{new},-}(\Gamma_0(m))$, then one has a lifting $F_f \in S_\kappa(\Gamma[m])$ due to Skoruppa-Zagier and Gritsenko.

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2. His construction is local in nature, so one can form many Saito-Kurokawa lifts from a single $f$.
3. In this talk we will give a precise statement about "mixed-level" Saito-Kurokawa lifts and outline a classical construction of such liftings.

# Mixed congruence level paramodular group $\Gamma_M[t]$

**Definition**

Let $t$ and $M \in \mathbb{N}$.

$$\Gamma_M[t] = \mathrm{Sp}_4(\mathbb{Q}) \bigcap \left\{ \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ M\mathbb{Z} & Mt\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ Mt\mathbb{Z} & Mt\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$

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2. For $M = 1$, $\Gamma_M[t] = \Gamma[t]$; hence we can say that the paramodular group $\Gamma[t]$ is of congruence level 1.
3. Similarly, for $t = 1$, the group $\Gamma_M[t]$ is the congruence subgroup $\Gamma_0^{(2)}(M)$ of $\mathrm{Sp}_4(\mathbb{Z})$.

# The Possible Saito-Kurokawa lifts

Let $t, M \in \mathbb{N}$ be odd square-free such that $\gcd(M, t) = 1$.
Fix $f$ a newform in $S^{\text{new}}_{2\kappa-2}(tM)$.

Depending on certain choices of a set $S$ of places $p$ with condition on the Atkin-Lehner eigenvalue at $p$, the possible Saito-Kurokawa lifts that $f$ can have are the following:

- If $f \in S^{\text{new},-}_{2\kappa-2}(\Gamma_0(Mt))$ then $F_f \in S_\kappa(\Gamma[tM])$.
- If $\kappa$ is even
  1. $F_f \in S_\kappa(\Gamma_M[t])$.
  2. $F_f \in S_\kappa(\Gamma_t[M])$.
  3. $F_f \in S_\kappa(\Gamma_0^{(2)}(tM))$.

# A couple definitions before the main theorem

Define

$$S^{t}_{2\kappa-2}(\Gamma_0(Mt)) = \{f \in S_{2\kappa-2}(\Gamma_0(Mt)) : f|W_t = (-1)^{\kappa} f\}.$$

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Recall that for $F \in S_\kappa(\Gamma)$ a Siegel eigenform, the Spinor $L$-function is defined by

$$L(s, F, \mathrm{spin}) = \zeta(2s - 2k + 4) \sum_{n \geq 1} \lambda_F(n) n^{-s}.$$

# Main Theorem

**Theorem**

*(Brown, Z.) Let $M$ and $t$ be odd square-free integers, $\gcd(M,t)=1$, $\kappa \geq 2$ an even integer, and $f \in S_{2\kappa-2}^{t,\mathrm{new}}(\Gamma_0(Mt))$ a newform. Let $\epsilon_p$ be the eigenvalue of $f$ under the Atkin-Lehner involution at $p$ and let $\eta_p$ be the Atkin-Lehner involution of degree 2 at $p$. There exists an eigenform $F_f \in S_\kappa(\Gamma_M[t])$, unique up to constant multiples, whose Spinor L-function is given by*

$$L(s,F_f,\mathrm{spin}) = \left( \prod_{\substack{p \mid M \\ \epsilon_p=-1}} (1-p^{-s+\kappa-1}) \right) \zeta(s-\kappa+1)\zeta(s-\kappa+2)L(s,f).$$

*Moreover, for each $p \mid t$ we have $\eta_p F_f = \epsilon_p F_f$ and for each $p \mid M$ we have $\eta_p F_f = F_f$.*

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2. We can construct the lifting without the requirement $M$ and $t$ be square-free. We only require square-free in order to get uniqueness (it won't be unique in general) as well as to get the correct $L$-functions.
3. We use representation theoretic methods to get uniqueness and the result on the $L$-functions, which is why we require odd and square-free.

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2. Next we lift from $S^{\text{Koh}}_{\kappa-1/2}(\Gamma_0(Mt))$ to $J^c_{\kappa,t}(\Gamma_0(M)^J)$.
3. Finally, we generalize Gritsenko's lifting to get a map $J^c_{\kappa,t}(\Gamma_0(M)^J)$ to $S_\kappa(\Gamma_M[t])$. This is the focus of the remainder of the talk.

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# Fourier Series of a Siegel Modular Form of degree 2

- For $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{h}^2$ we write it as a row vector $(\tau, z, \tau')$, $\tau, \tau' \in \mathfrak{h}^1, z \in \mathbb{C}$, and $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\,\mathrm{Im}(\tau')$.

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- We write $F(\tau, z, \tau')$ instead of $F(Z)$.
- For every $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \mathbb{S}_2^{\geq 0}(\mathbb{Z})$ , we write $a_F(n, r, m)$ for $a_F(T)$ where $n, r, m \in \mathbb{Z}, n, m \geq 0$ and $r^2 \leq 4mn$.

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The Fourier expansion of $F \in S_\kappa(\Gamma)$ takes the form

$$F(\tau, z, \tau') = \sum_{\substack{m,n,r \in \mathbb{Z} \\ m,n,4mn - r^2 > 0}} a_F(n, r, m) e(n\tau + rz + m\tau').$$

# Fourier-Jacobi expansion of Mixed Congruence level Paramodular forms

Let $F \in M_\kappa(\Gamma_M[t])$. We can rewrite its Fourier expansion as

$$F(\tau, z, \tau') = \sum_{m \geq 0} \phi_{mt}(\tau, z) e(2\pi i(mt)\tau').$$

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**Corollary**

*Let $F \in M_\kappa(\Gamma[t])$. For each $m$, $F$'s Fourier-Jacobi coefficient $\phi_{mt}$ belongs to $J_{\kappa,mt}$.*

**Theorem (Z.)**

*Let $\phi_t$ be a Jacobi cusp form of weight $\kappa \geq 2$, index $t$, and level $\Gamma_0(M)^J$ with Fourier expansion*

$$\phi_t(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, n \geq 0 \\ 4nt > r^2}} c(n, r) e(2\pi i(n\tau + rz)).$$

*Then*

$$\mathcal{G}_M(\phi_t)(\tau, z, \tau') := \sum_{m \geq 1} V_m(\phi_t) e(2\pi i m t \tau')$$

*lies in $S_\kappa(\Gamma_M[t])$ where $V_m$ is the index-shifting operator.*

# Remarks

1. For $t = 1$ the lifting

$$\mathcal{G}_M : \phi_1 \to \mathcal{G}_M(\phi_1)$$

is the Maass lifting with level $M$

$$\mathcal{V} : J^c_{\kappa,1}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_0^{(2)}(M)).$$

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is the Maass lifting with level $M$

$$\mathcal{V} : J^c_{\kappa,1}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_0^{(2)}(M)).$$

2. For $M = 1$ the lifting

$$\mathcal{G}_M : \phi_t \to \mathcal{G}_M(\phi_t)$$

is Gritsenko's lifting

$$\mathcal{G} : J^c_{\kappa,t} \to S_\kappa(\Gamma[t]).$$

# Families of liftings

Let $F \in S_\kappa(\Gamma_M[t])$ have Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m \geq 1} \phi_{mt}(\tau, z) e(2\pi i m t \tau').$$

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One obtains an infinite family of liftings given by

$$S_\kappa(\Gamma_M[t]) \longrightarrow \prod_{m\geq 1} J^c_{\kappa, mt}(\Gamma_0(M)^J) \longrightarrow \prod_{m \geq 1} S_\kappa(\Gamma_M[mt])$$

$$F \longmapsto (\phi_{mt})_{m\geq 1} \longmapsto (\mathcal{G}(\phi_{mt}))_{m \geq 1}.$$

One can iterate this process indefinitely.

**Corollary (Z.)**

*The map* $\mathcal{G}_M : J^c_{\kappa,t}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t])$ *is injective.*

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The composition

$$J^c_{\kappa,t}(\Gamma_0(M)^J) \to S_\kappa(\Gamma_M[t]) \to \prod_{m \in \mathbb{N}} J^c_{\kappa,mt}(\Gamma_0(M)^J) \to J^c_{\kappa,t}(\Gamma_0(M)^J)$$

is the identity.

# Image of the Mixed Level Lifting

The image of the lifting $\mathcal{G}_M$ is the subspace of $S_\kappa(\Gamma_M[t])$ consisting of modular forms whose Fourier coefficients satisfy the following relations

$$a(n, r, mt) = \sum_{\substack{d|(n,r,m) \\ r^2<4nmt \\ (d,M)=1}} d^{\kappa-1} c\left(\frac{nm}{d^2}, \frac{r}{d}\right).$$

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**Corollary (Z.)**

*We have the following isomorphism of vector spaces*

$$J_{\kappa,t}^c(\Gamma_0(M)^J) \cong S_\kappa^*(\Gamma_M[t]).$$

**Lemma (Z.)**

*The Mixed level lifting $\mathcal{G}_M$ is Hecke equivariant with respect to the Hecke algebra homomorphism $\iota : \mathbb{T}_{\mathbb{Z}}^{S,tM} \to \mathbb{T}_{\mathbb{Z}}^{J,tM}$ given by*

$$\iota(T_S(p)) = -T_J(p) + p^{k-1} + p^{k-2} \quad (p \nmid tM),$$
$$\iota(T'_S(p)) = (p^{k-1} + p^{k-2})T_J(p) + 2p^{2k-3} + p^{2k-4} \quad (p \nmid tM).$$

Equivalently, the lifting $\mathcal{G}_M$ satisfies

$$(\mathcal{G}_M(\phi))|T = \mathcal{G}_M(\phi|\iota(T))$$

for any $T \in \mathbb{T}_{\mathbb{Z}}^{S,tM}$.

# Some future directions

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3. Construct congruences between the lifted forms and non-lifted Siegel forms. This has applications to a further conjecture of Brumer-Kramer. Namely, if $M = 1$, they conjecture a congruence modulo $p$ between a Saito-Kurokawa lift and a non-lifted form should correspond to a $p$-torsion point on the abelian surface given by the non-lifted form. So far their congruences are all computational.