Modular forms and elliptic curves over the cubic field of discriminant -23

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(joint work with P. Gunnells)

Overview

Let F be a number field.

- E/F elliptic curve

  $\updownarrow$?

- cuspidal Hecke eigenform f on GL_2/F whose eigenvalues are closely related to the number of points on E over various finite fields

We investigate the case when F is the complex cubic field of discriminant -23.

Modular forms via cohomology

Borel conjectured and Franke proved that the complex cohomology of an arithmetic group Γ can be computed in terms of certain automorphic forms.

1. Construct a suitable topological space X.
2. Compute cohomology.
3. Compute Hecke action on cohomology classes.

Cohomology

For Γ torsion-free, the quotient $\Gamma\backslash X$ is a Eilenberg-Mac Lane space.

$$H^*(\Gamma;\mathbb{C}) \simeq H^*(\Gamma\backslash X;\mathbb{C}).$$

These are the cohomology spaces that are built from certain automorphic forms.

- Replace $\mathbb{C}$ with complex representation of $\mathbf{G}(\mathbb{Q})$ to introduce weight structure.
- Isomorphism is true even if Γ has torsion.

Motivating example

$$\begin{array}{rcl} \mathbf{G} & = & \mathrm{SL}_2 \\ G & = & \mathrm{SL}_2(\mathbb{R}) \\ X & = & \text{complex upper half-plane } \mathfrak{h} \\ \Gamma & = & \Gamma_0(N) \text{ congruence subgroup of matrices that} \\ & & \text{are upper triangular modulo } N. \end{array}$$

Modular curve $\Gamma\backslash\mathfrak{h}$

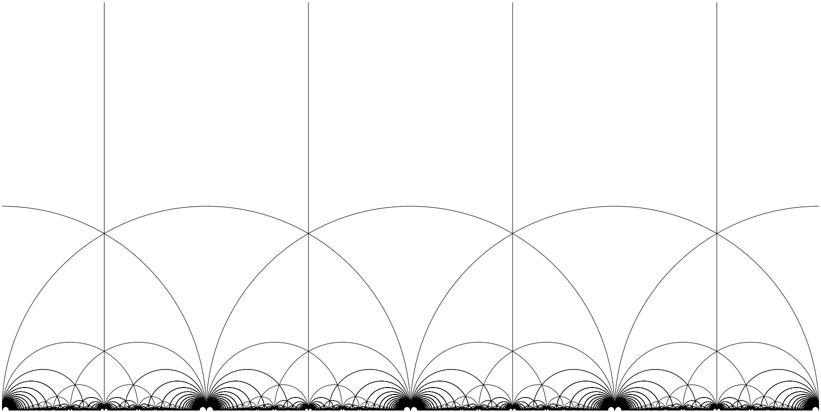

Figure: Fundamental domain for action of $SL_2(\mathbb{Z})$.

Eichler-Shimura isomorphism

We have

$$H^1(\Gamma\backslash X;\mathbb{C}) \simeq S_2(N) \oplus \overline{S}_2(N) \oplus \mathrm{Eis}_2(N),$$

where

- S_2 is space of weight 2 cusp forms,
- Eis_2 is space of weight 2 Eisenstein series,

Tessellation by ideal triangles

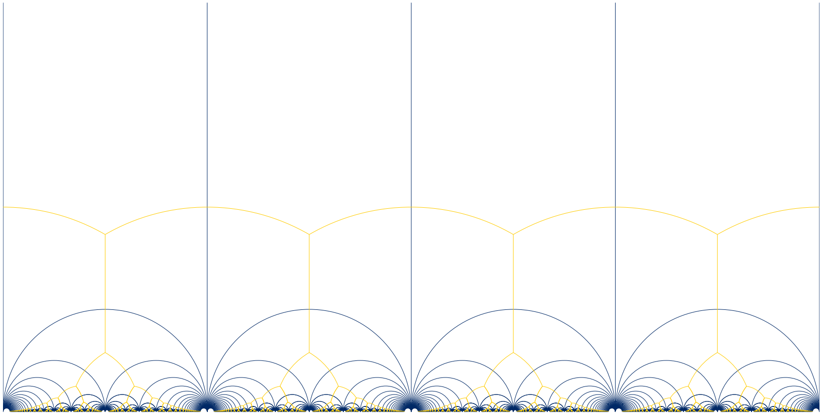

Figure: Well-rounded tree (gold) and dual tessellation (blue).

Modular symbols

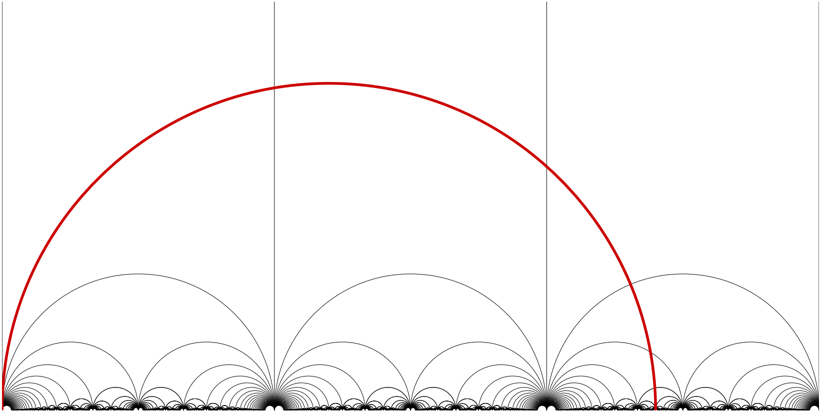

Figure: Unimodular symbols (black) and non-reduced symbol (red).

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Reduction of modular symbol

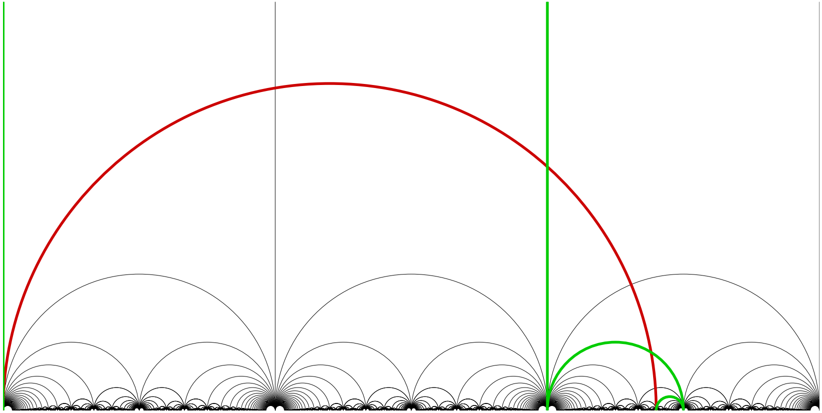


Figure: Re-expression of non-reduced symbol (red) as a sum of unimodular symbols (green).

General setting

Let F be a number field of class number 1 with ring of integers $\mathcal{O}$.

$$
\begin{aligned}
\mathbf{G} &= \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n \\
G &= \mathrm{GL}_n(S) \simeq \left(\prod \mathrm{GL}_n(\mathbb{R})\right) \times \left(\prod \mathrm{GL}_n(\mathbb{C})\right) \\
X &= \text{associated symmetric space } G/KA_{\mathbf{G}} \\
&= \text{space of (Hermitian) forms modulo homothety} \\
\Gamma &\subseteq \mathrm{GL}_n(\mathcal{O}), \text{ congruence subgroup}
\end{aligned}
$$

Which n and F?

- $F = \mathbb{Q}$

$n = 3$: Ash-Grayson-Green, Ash-McConnell, van Geemen-van der Kallen-Top-Verberkmoes

$n = 4$: Ash-Gunnells-McConnell

- F a complex quadratic field

$n = 2$: Grunewald, Cremona, Schwermer, Vogtmann, Y

- F a totally real field

$n = 2$: Dembélé, Voight

- F the CM quartic field $\mathbb{Q}(\zeta_5)$

$n = 2$: Gunnells-Hajir-Y

Setting

Let F be the (mixed signature) cubic field of discriminant -23 defined by polynomial $x^3 - x^2 + 1$.

$$
\begin{aligned}
G &\simeq \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{C}) \\
V &= \mathrm{Sym}_2(\mathbb{R}) \times \mathrm{Herm}_2(\mathbb{C}) \\
C &= \mathrm{Sym}_2(\mathbb{R})^+ \times \mathrm{Herm}_2(\mathbb{C})^+ \\
X &= C/\sim \quad \text{6-dimensional symmetric space}
\end{aligned}
$$

Natural G-action on C descends to G-action on X.

Modular forms and elliptic curves

For each cuspidal Hecke eigenform with integral eigenvalues $a_{\mathfrak{p}}$, one wants E/F such that

$$|E(\mathbb{F}_{\mathfrak{p}})| = \mathrm{N}(\mathfrak{p}) + 1 - a_{\mathfrak{p}}.$$

Voronoï polyhedron

View point $x \in \mathcal{O}^2$ as point $q(x) \in \bar{C}$

$$q(x)_v = x_v x_v^*.$$

- $q(\mathcal{O}^2)$ is discrete in $\bar{C}$.
- Voronoï Π is convex hull of $\{q(x) \mid \mathcal{O}^2 \setminus 0\}$.

Cell structure of Π descends to tessellation of X.

6-dimensional cells

There are nine $GL_2(\mathcal{O})$-classes of 6-polytopes.

- Seven are simplicial with f-vector $(7, 21, 35, 35, 21, 7)$.
- One has f-vector $(8, 28, 56, 68, 48, 16)$.
- One has f-vector $(9, 36, 81, 108, 81, 27)$.

Voronoi cells for F

Dimension	# of classes
6	9
5	35
4	47
3	31
2	10
1	2

The number of $GL_2(\mathcal{O})$-classes of Voronoï cells are given. The 2-dimensional cells are simplicial.

Sharbly complex

The *sharbly complex* is a resolution of the Steinberg module that can be used to compute the cohomology (Ash, Lee-Szczarba).

$$H^{\nu-k}(\Gamma;\mathbb{C}) \simeq H_k(S_*(\Gamma))$$

Sharbly complex

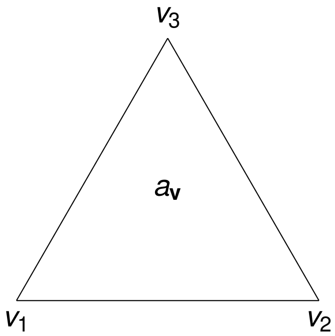
Let S_k, $k \geq 0$, be the Γ-module A_k/C_k, where A_k is the set of formal $\mathbb{C}$-linear sums of symbols $[v] = [v_1, \cdots, v_{k+n}]$, where each v_i is in F^n, and C_k is the submodule generated by

1. $[v_{\sigma(1)}, \cdots, v_{\sigma(k+2)}] - \mathrm{sgn}(\sigma)[v_1, \cdots, v_{k+n}]$,
2. $[v, v_2, \cdots, v_{k+2}] - [w, v_2, \cdots v_{k+n}]$ if $Ray(vv^*) = Ray(ww^*)$, and
3. $[\mathbf{v}]$, if $\mathbf{v}$ is *degenerate*, i.e., if $v_1, \cdots, v_{k+n}$ are contained in a hyperplane,

with the usual boundary map.

$$H^{\nu-k}(\Gamma; \mathbb{C}) \simeq H_k(S_*(\Gamma))$$

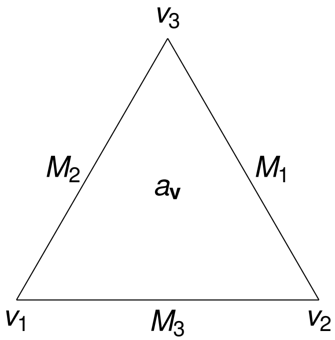
1-sharblies



For $n = 2$, 1-sharblies are formal sums of triples of vertices of Π

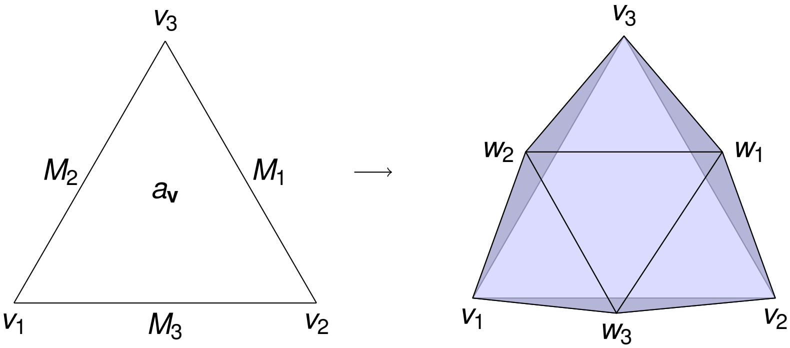
$$\sum a_{\mathbf{v}}[v_1, v_2, v_3].$$

1-sharbly cycle

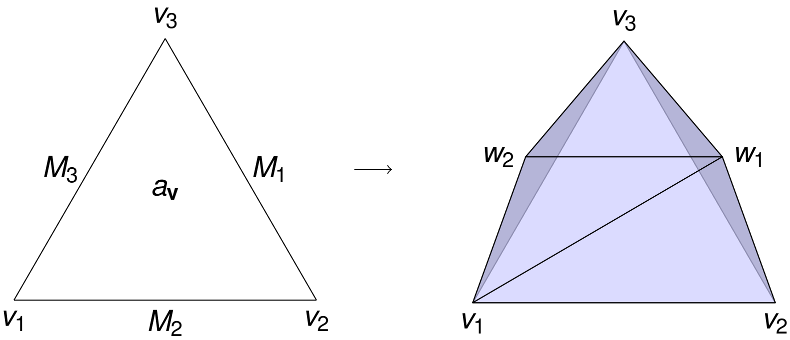


- $a_{\mathbf{v}}$ = coefficient
- v_i = vertices
- M_i = lift data
- boundary vanishes modulo Γ

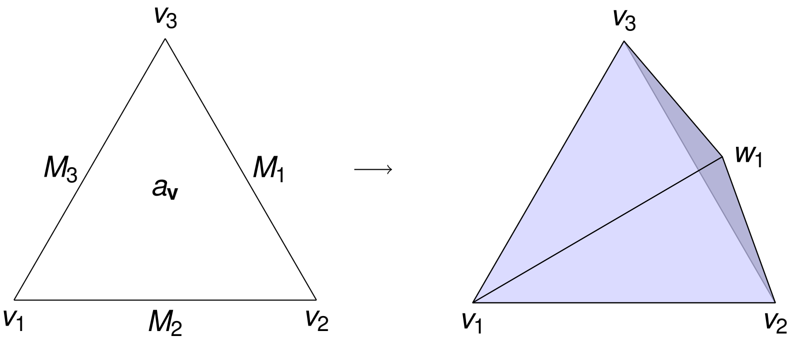
Generic case (3 bad edges)



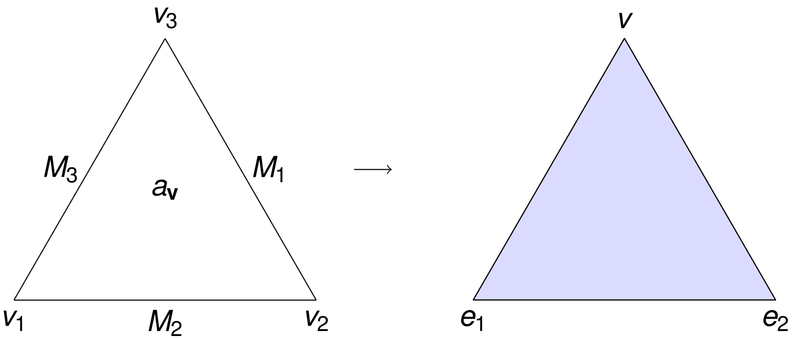
Less generic case (2 bad)



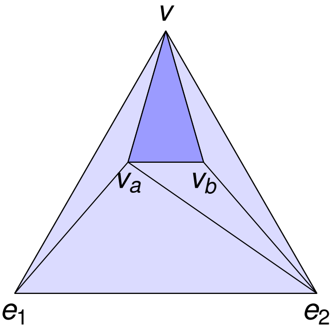
Even less generic case (1 bad)

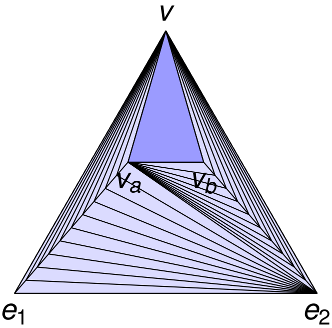


Very special case (0 bad)



Very special case (0 bad)





Example

For level $(4t^2 - t - 5)$ of norm 89, the cuspidal space is 1-dimensional. We find an elliptic curve

$$[a_1, a_2, a_3, a_4, a_6] = [t - 1, -t^2 - 1, t^2 - t, t^2, 0]$$

with matching Hecke data.

More examples . . .

Nm	a_1	a_2	a_3	a_4	a_6
89	$t-1$	$-t^2-1$	t^2-t	t^2	0
107	0	$-t$	$-t-1$	$-t^2-t$	0
115	$-t^2+t-1$	$-t^2+1$	$t-1$	-1	$-t^2$
136	$-t^2$	-1	$-t^2+1$	$t+1$	0
161	t^2-t-1	$-t^2+t-1$	t^2-t+1	t^2-t	$t-1$
167	t^2+1	$t+1$	t^2+t-1	$-t^2-t+1$	$-t^2+t+1$
185	t	$-t^2+t+1$	$t+1$	0	0
223	1	t^2	t^2+t-1	$-t^2+t-1$	1
253	-1	$-t^2-t$	$-t^2-t$	$-t^2-t$	0
259	0	1	$-t^2-t-1$	t^2-t+1	$-t^2-t+1$
275	$-t^2+t$	t	t^2-t	0	0
289	-1	t^2-t	t	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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Thank you.