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Conjectural Extensions of the Gross-Zagier Theorem

Gross Zagier Theorem:

$K \subset \mathbb{C}$ imag. quad. For simplicity assume class number 1.
$D = |\mathrm{disc}(K)|$.
$N \in \mathbb{Z}^+$ s.t. $p \mid N \Rightarrow p$ split in K (Heegner hyp)
Fix $\mathfrak{n} \subset \mathcal{O}_K$ s.t. $\mathcal{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$.
Heegner pt.

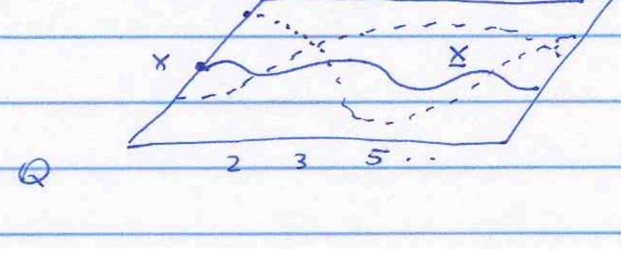
$$\begin{array}{ccc} x = \left[\mathbb{C}/\mathcal{O}_K \to \mathbb{C}/\mathfrak{n}^{-1}\right] \in X_0(N)(K) & & \infty \\ \downarrow & & \downarrow \\ x \in J_0(N)(K) & & 0 \end{array}$$

Let $f \in S_2^{\mathrm{New}}(\Gamma_0(N))$.
idempotent $e_f \in$ Hecke algebra s.t. $e_f \cdot S_2(\Gamma_0(N)) = \mathbb{C} \cdot f$.

$x_f = e_f \cdot x \in J_0(N)(K) \otimes_{\mathbb{Z}} \mathbb{C}$.

<u>Thm (Gross-Zagier):</u> $\langle x_f, x_f \rangle_{N\text{-}T} = \dfrac{1}{\langle f, f \rangle_{Pet}} L'(f, K, 1)$.

<u>Step 1:</u> Integral model $\underline{X_0(N)} \to \mathrm{Spec}(\mathcal{O}_K)$
$\underline{x} \in \underline{X_0(N)}(\mathcal{O}_K)$ Zariski closure of x.



$\underline{X_0(N)}$

dotted = $\underline{T_m x}$

$I(\underline{x}, \underline{T_m x})$ Gross computes intersection numbers.

$a_m(\Theta(\tau)\, E'(\tau,1))$ Fourier coeffs computed by Zagier.

$\Theta(\tau) = \sum_{m=0}^{\infty} \#\{a \in \mathcal{O}_K : a\bar{a} = m\}\, q^m \in M_1(\Gamma_0(D), \chi)$; $\chi = \chi_{K/\mathbb{Q}}$

$E(\tau, s)$ is a wt 1 Eisenstein series.

They show these are equal. Of course there are a lot of other things going on here.

Step II: $\Rightarrow \quad \sum_m \langle x, T_m x\rangle_{NT} \cdot q^m = \Theta(\tau)\, E'(\tau,1).$

Apply e_f to both sides:

$$\langle x_f, x_f\rangle_{NT}\, f = \frac{\langle f, \Theta E'\rangle_{Pet}}{\langle f, f\rangle_{Pet}}\, f$$

Now use Rankin-Selberg:

$$\langle f, \underbrace{\Theta E'}_{\Theta(\tau) E(\tau, s)}\rangle_{Pet} = L(f, K, s).$$

One side is geometric, one side is automorphic. To generalize, we must consider how to generalize each side. The automorphic form side is easier to generalize.

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Fix $n \geq 2$. $f \in S_n^{new}(\Gamma_0(D), \chi^n)$.

Let (Λ, H) be a self-dual pos. def. Hermitian lattice of rank $n-1$.

- Λ is proj. $\mathcal{O}_K$-module of rank $n-1$
- $H : \Lambda \times \Lambda \to \mathcal{O}_K$ pos. def. Hermitian form inducing
  $\Lambda \cong \mathrm{Hom}_{\mathcal{O}_K}(\Lambda, \mathcal{O}_K)$.

$$\Theta_\Lambda(\tau) = \sum_{m=0}^{\infty} \#\{ a \in \Lambda : H(a,a) = m \} q^m \in M_{n-1}(\Gamma_0(D), \chi^{n-1})$$

<u>Remark:</u> $(\Lambda, H) = (\mathcal{O}_K, Nm)^{n-1}$ gives $\Theta_\Lambda = \Theta^{n-1}$.

Rankin-Selberg L-function:

$$L(f \times \Theta_\Lambda, s) = \sum_{m=1}^{\infty} \frac{a_m(f)\, a_m(\Theta_\Lambda)}{m^{\boxed{s}}}$$

shift so funct. eqn is so

$s \longleftrightarrow$ ~~...~~ $-s$.

$L'(f \times \Theta_\Lambda, 0) = ????$

This gives the automorphic side of things.

Now for the geometric side. Zhang and Nekovar have considered such things for Θ wt 1. The geometry then is with Kuga-Sato varieties. In this case things are totally different.

$GU(n-1,1)$ Shimura variety.

$M = M_{(n-1,1)}$ = moduli space of (A, κ, λ, F) over $\mathcal{O}_K$-algebras $\iota: \mathcal{O}_K \to R$

$\downarrow$

$\mathrm{Spec}(\mathcal{O}_K)$

abs. dim = n
regular, flat

- $A/_R$ abelian scheme of dim n.
- $\kappa: \mathcal{O}_K \to \mathrm{End}(A)$ is an action of $\mathcal{O}_K$
- $\lambda: A \to A^\vee$ $\mathcal{O}_K$-linear principal polarization.
- $F \subset \mathrm{Lie}(A)$ $\mathcal{O}_K$-stable direct R-module summand of rank $n-1$ s.t. $\mathcal{O}_K$ acts on F via $\iota: \mathcal{O}_K \to R$

  $\mathcal{O}_K$ acts on $\mathrm{Lie}(A)/F$ via conjugate of

Has a canonical toroidal compactification.

<u>Remark:</u> $M_{(1,1)}$ is almost $X_0(D)$.

$$X_0(D) \longrightarrow M_{(1,1)}$$

$$[E \xrightarrow{\phi} E'] \longmapsto E \times E' \text{ with } \sqrt{-D} \text{ acting as}$$

$$(x,y) \mapsto (\phi^\vee(y), -\phi(x)).$$

M has a canonical point: $\mathbb{E} = \mathbb{C}/\mathcal{O}_K$ with obvious $\mathcal{O}_K$-action

$\overline{\mathbb{E}} = \mathbb{E}$ with complex conj. action.

$\mathbb{E}^{n-1} \times \overline{\mathbb{E}} \in M(\mathcal{O}_K)$.

From (Λ, H), we can construct $\mathbb{B} = (\Lambda \otimes_{\mathcal{O}_K} \mathbb{C})/\Lambda$

symplectic form $\lambda: \Lambda \times \Lambda \to \mathbb{Z}$ $\quad \lambda(x,y) = \dfrac{H(x,y) - H(y,x)}{\sqrt{-D}}$.

This defines polarization, so gives an abelian variety.

Heegner point $x \in \mathbb{B} \times \overline{\mathbb{E}} \in M(\mathcal{O}_K)$.

Kudla - Rapoport Divisors:

Fix $m > 0$. Given $A \in \mathcal{M}(R)$ $\qquad R = \mathcal{O}_K$-algebra.

$\mathcal{O}_K$-module $\quad \mathrm{Hom}_{\mathcal{O}_K}(\mathbb{E}/_R, A)$ $\quad$ has a positive def.
Hermitian form.

$$\langle f_1, f_2 \rangle = f_2^\vee \circ f_1 \in \mathrm{End}_{\mathcal{O}_K}(\mathbb{E}/_R) = \mathcal{O}_K.$$

$KR(m) =$ moduli of $(A, f)/_R$ where

- $A \in \mathcal{M}(R)$
- $f \in \mathrm{Hom}_{\mathcal{O}_K}(\mathbb{E}/_R, A)$ s.t. $f^\vee \circ f = m$

$$KR(m) \longrightarrow \mathcal{M}$$

<u>Remark:</u> 1) $KR(1) =$ locus of A on $\mathcal{M}$ at which $A \cong ? \times \mathbb{E}$.

2) These divisors come from $GU(n-2,1) \hookrightarrow GU(n-1,1)$.

Idea (too naive):

$$\begin{array}{ccc} & \nearrow & \mathrm{Pic}^0(M) \\ M & & \updownarrow \text{ dual abelian varieties.} \\ & \searrow & \mathrm{Alb}(M) \end{array}$$

$$\langle KR(1)_f, x_f \rangle_{NT} = L'(f \times \Theta_\Lambda, 0).$$

There are problems with this though.

Arithmetic Chow groups:

$\hat{Z}^1(\mathcal{M})$ ~~[illegible]~~ arithmetic divisors

$\hat{Z}^1(M)$ = $\mathbb{R}$-linear combinations (D, G) where D is a divisor on M, G is Green function for D
i.e, if $f = 0$ is local equation for $D(\mathbb{C})$ in $M(\mathbb{C})$ then

$$G(z_1) + \log|f|^2 \text{ extends smoothly}$$

across $D(\mathbb{C})$.

Principal arithmetic divisors $\quad \hat{B}^1(M) = \mathbb{R}$-span of $(\mathrm{div}(f), -\log|f|^2)$ for all rational functions f.

Arithmetic Chow group $\quad \widehat{CH}^1(M) = \hat{Z}^1(M) / \hat{B}^1(M)$.

$\exists$ linear functional "intersection with x"

$$\widehat{\deg}_x : \widehat{CH}^1(M) \to \mathbb{R}.$$

<u>Fact:</u> There are natural Green functions G_m for $KR(m)$

$$\widehat{KR}(m) = (KR(m), G_m) \in \widehat{CH}^1(M).$$

<u>Conjecture:</u> ① For some choice of $\widehat{KR}(0) \in \widehat{CH}^1(M)$ the generating series

$$\widehat{KR} = \sum_{m=0}^{\infty} \widehat{KR}(m) q^m \in \widehat{CH}^1(M)[[q]]$$

lies in

$$M_n(\Gamma_0(N), \chi^n) \otimes_{\mathbb{R}} \widehat{CH}^1(M)$$

② Given $f \in S_n^{new}(\Gamma_0(N), \chi^n)$ can form

$$\hat{Z}(f) = \langle f, \widehat{KR} \rangle_{Pet} \in \widehat{CH}^1(M)$$

Then $\widehat{\deg}_X \hat{Z}(f) = L'(f \times \Theta_\Lambda, 0)$.

<u>Sketch of partial proof:</u>

Find wt 1 Eisenstein series $E(\tau, s)$ s.t.

① $L(f \times \Theta_\Lambda, s) = \langle f, \Theta_\Lambda \cdot E' \rangle_{Pet}$.

② $\forall m \quad \widehat{\deg}_X \widehat{KR}(m) = a_m(\Theta_\Lambda E')$.

Suppose one knew these. Then one computes

$$\begin{aligned}\widehat{\deg}_X \hat{Z}(f) &= \widehat{\deg}_X \langle f, \widehat{KR} \rangle_{Pet} \\ &= \langle f, \widehat{\deg}_X \widehat{KR} \rangle_{Pet} \\ &= \langle f, \Theta_\Lambda \cdot E' \rangle_{Pet} && \text{by ②} \\ &= L'(f \times \Theta_\Lambda, 0) && \text{by ①}.\end{aligned}$$

So all the hard work goes into proving ②.

<u>Prop:</u> Suppose $m > 0$ and (Λ, H) does not represent m.
Then ② is true.

This simple case is the evidence for this conjecture. The Eisenstein series $E(\tau, s)$ is given in a paper of Kudla-Yang.

Let $K = \mathbb{Q}(i) \qquad (\Lambda, H) = (\mathbb{Z}[i], Nm) \oplus (\mathbb{Z}[i], Nm)$.
This represents every integer.. So this shows the ~~prop~~ condition in the prop. is very restrictive.