Ternary Quadratic Forms and Half-Integral Weight Modular Forms

Alia Hamieh

University of British Columbia

PANTS 17, Dec. 3, 2011

Outline

Half-Integer Weight Modular Forms

In order to define a modular form of half-integeral weight $\frac{k}{2}$ for a congruence subgroup Γ' one would consider a holomorphic function f on $\mathbb{H}$ satisfying $f(\gamma z) = (cz+d)^{\frac{k}{2}} f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$.

However, $J(\gamma, z) = (cz+d)^{\frac{k}{2}}$ is not an automorphy factor because it does not satisfy the identity

$$J(\alpha\beta, z) = J(\alpha, \beta z) \times J(\beta, z) \qquad \text{for all } \alpha, \beta \in \Gamma',\ z \in \mathbb{H}$$

The natural way out of this complication is to define the automorphy factor to be the k-th power of

$$j(\gamma, z) = \left(\frac{c}{d}\right) {\epsilon_d}^{-1} \sqrt{cz+d} \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

where

$$\left(\frac{c}{d}\right) = \begin{cases} -\left(\frac{c}{|d|}\right), & \text{if } c < 0, d < 0, \\ \\ \left(\frac{c}{|d|}\right), & \text{otherwise.} \end{cases}$$

and

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4; \\ i, & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

For a congruence subgroup $\Gamma' \in \Gamma_0(4)$ define,

$$f(z)|[\gamma]_{\frac{k}{2}} = j(\gamma, z)^{-k} f(\gamma z)$$

Definition

Let k be an odd integer and let Γ' be a congruence subgroup such that $\Gamma' \subset \Gamma_0(4)$. We say that $f(z)$ is a modular form of weight $\frac{k}{2}$ for Γ' and write $f \in M_{\frac{k}{2}}(\Gamma')$ if:

- f is invariant under $[\gamma]_{\frac{k}{2}}$ for all $\gamma \in \Gamma'$
- f is holomorphic as function on the upper half plane $\mathbb{H}$
- f is holomorphic at every cusp

If f vanishes at every cusp we say that it is a cusp form and write $f \in S_{\frac{k}{2}}(\Gamma')$

Another important definition is that of the space of modular forms with a character χ defined modulo $4N$. This is denoted by $M_{\frac{k}{2}}(4N, \chi)$ and defined to be the set

$$\{f \in M_{\frac{k}{2}}(\Gamma_1(4N)) : f|[\gamma]_{\frac{k}{2}} = \chi(d)f \ \forall \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)\}$$

$$S_{\frac{k}{2}}(4N, \chi) = M_{\frac{k}{2}}(4N, \chi) \cap S_{\frac{k}{2}}(\Gamma_1(4N))$$

Moreover,

$$M_{\frac{k}{2}}(\Gamma_1(4N)) = \bigoplus_{\chi} M_{\frac{k}{2}}(4N, \chi)$$

Hecke Operators

Given $f = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \in M_{\frac{k}{2}}(4N, \chi)$:

- We can have non-trivial operators T_m only for square m or for $(m, 4N) \neq 1$.
- $$T_{p^2}(f)(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$$

  where

  $$b_n = a_{p^2 n} + \chi(p)(\frac{(-1)^{\frac{k-1}{2}} n}{p}) p^{\frac{k-3}{2}} a_n + \chi(p^2) p^{k-2} a_{\frac{n}{p^2}}.$$
- $$T_p(f)(z) = \sum_{n=0}^{\infty} a_{np} e^{2\pi i n z} \qquad \text{when } p|N.$$

Theorem (Shimura(1973))

Let $k \geq 3$ be an odd integer. Let $f \in S_{\frac{k}{2}}(4N, \chi)$ be a common eigenfunction for all T_{p^2} with λ_p being the corresponding eigenvalue. Define the sequence of complex numbers $\{b_n\}$ by the formal identity

$$\sum_{n=1}^{\infty} b_n n^{-s} = \prod_p \frac{1}{1 - \lambda_p p^{-s} + \chi(p^2) p^{k-2-2s}}$$

Then $g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$ belongs to $M_{k-1}(N', \chi^2)$ for some integer N' which is divisible by he conductor of χ^2. If $k \geq 5$, we get a cusp form.

Given $F \in S_{k-1}^{new}(\chi^2)$ a newform such that $T_p F = b_p F$ for all p, define

$S_{\frac{k}{2}}(4N, \chi, F) = \{g \in S_{\frac{k}{2}}(4N, \chi) : T_{p^2} g = b_p g$ for almost all $p \not| 4N\}$.

Shimura proved

$$S_{\frac{k}{2}}(4N, \chi) = \bigoplus_F S_{\frac{k}{2}}(4N, \chi, F)$$

where the sum is over a finite number of newforms F of weight $k-1$, character χ^2 and level $M(F)$ dividing $2N$.

Description of Waldspurger's Result

Waldspurger proved that, under quite general conditions on F, N and χ, there exists a basis for $S_{\frac{k}{2}}(4N, \chi, F)$ such that for every positive integer n the Fourier coefficient $a_n(g_i)$ of a basis element g_i is the product of two factors:

- a product of local terms $c_i(n, F)$ each of which is completely determined by the local components of F and χ according to explicit formulae.
- a global factor $A_F(n)$ whose square is the central critical value of the L-function of the newform F twisted by a quadratic character depending on n.

- Given a newform $F \in S_{k-1}^{new}(\Gamma_0(N))$ that satisfies $L(F, \frac{k-1}{2}) \neq 0$, we compute a basis for $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ where N is odd square-free integer and $k \equiv 3 \mod 4$.
- In the light of Waldspurger's work, our task is reduced to computing the global factors $A_F(n)$.
- We construct of a half-integral weight modular form $g \in S_{\frac{k}{2}}^{+}(\Gamma_0(4N), F)$.
- We express the globlal factors $A_F(n)$ in terms of the Fourier coefficients of g.

Basic Setup

- For each $p|N$, $W_pF = w_pF$.
- Let $S = \{q_1, ..., q_r\}$ be the set of all prime factors q of N such that $w_q = -1$
- Let B be the definite quaternion algebra over $\mathbb{Q}$ ramified at S and ∞, and O be an Eichler order of square-free level N in B.
- Let $\bar{C} = \{[I_1], [I_2], ..., [I_H]\}$ be the set of left O-ideal classes and X be the free abelian group over $\bar{C}$
- There is an action of Hecke operators on the $X(\mathbb{R}) = X \otimes_{\mathbb{Z}} \mathbb{R}$ denoted by t_n.
- $X(\mathbb{R})$ has an orthogonal basis of eigenvectors for $\{t_n\}$.

- For every left O-ideal I_i in C, let O_i be its right order, and let R_i be the subgroup of trace zero elements in the suborder $\mathbb{Z} + 2O_i$.
- The ternary quadratic form $N(x)$ for $x \in R_i$ is a positive definite integral quadratic form with level $4N$ and a square discriminant.
- For every left O-ideal class $[I_i]$, we associate the ternary theta series

$$g([I_i]) = \frac{1}{2} \sum_{x \in R_i} q^{N(x)} = \frac{1}{2} \sum_{D \geq 0} a_D([I_i]) q^D,$$

  then extend this association by linearity to $X(\mathbb{R})$.
- These modular forms have weight $\frac{3}{2}$, level $4N$ and trivial character
- Computing $g([I_i])$ up to a precision T amounts to computing the number of times $N(x)$ represents $1, 2, 3, ..., T$ as x varies over R_i. Therefore, it takes time roughly proportional to $T^{\frac{3}{2}}$

Proposition

For all $I \in X(\mathbb{R})$ and all $p \nmid 4N$ we have

$$T_{p^2}(g(I)) = g(t_p(I)).$$

Hence, if I_F is a non-zero element in the F-isotypical component of $X(\mathbb{R})$, then $g(I_F) \in S^+_{\frac{3}{2}}(\Gamma_0(4N), F)$

Theorem (Bocherer and Schulze-Pillot(1994))

The half-integral weight modular form $g = g(I_F) = \sum_{D \geq 1} c_D q^D$ *is non-zero if and only if* $L(F, 1) \neq 0$*. Moreover, if* $-D < 0$ *is a fundamental discriminant with* $\chi_{-D}(-N) \neq -1$*, then*

$$D^{\frac{1}{2}} L(F, 1) L(F, -D, 1) = 2^{\nu+2} c(F) c_D^2,$$

where $c(F)$ *is a complex constant depending only on* F *and*

$$2^{-\nu} = \prod_{q \mid \frac{N}{(N,D)}} \left(1 + \left(\frac{-D}{q} \right) w_q \right)$$

Some Notation

Let k, N, M, χ, χ_0, χ_t and ρ be given as follows:

- $k \geq 3$ written as $2\lambda + 1$
- N positive integer
- χ even Dirichlet character modulo $4N$
- χ_0 Dirichlet character defined by $\chi_0(n) = \chi(n) \left(\frac{-1}{n}\right)^\lambda$
- χ_t the quadratic character associated with the extension $\mathbb{Q}(\sqrt{t})/\mathbb{Q}$
- ρ irreducible automorphic representation of $GL_2(\mathbb{A})$ associated to F

Let F be a newform in $S_{k-1}^{new}(\chi^2)$ such that $S_{\frac{k}{2}}(L, \chi, F) \neq 0$ for some integer $L \geq 1$.
We also require that one of the following conditions is satisfied:

- The level of F is divisible by 16
- The conductor of χ_0 is divisible by 16
- ρ_2 is not supercuspidal

Theorem (Waldspurger(1981))

Given a newform F satisfying the above hypotheses, there exists a complex-valued function A_F on the set of square-free integers, $\mathbb{N}^{sq}$, such that

1. $(A_F(t))^2 = L(F \otimes \chi_0^{-1}\chi_t, \frac{1}{2})\epsilon(\chi_0^{-1}\chi_t, \frac{1}{2})$
2. $S_{\frac{k}{2}}(4N, F, \chi) = \bigoplus U(E, F, A_F)$*; the direct sum being taken over all integers E such that $N(F)|E|4N$*

Given an integer $E \geq 1$ and a prime number p, Waldspurger defined an integer n_p and a set $U_p(v_p(E), F)$ of finitely many functions c_p defined over $\mathbb{Q}_p^*$. The set $U_p(v_p(E), F)$ is a singleton and $n_p = 0$ for all but finitely many p. For every element $c_E = (c_{E,p}) \in U(E,F) = \prod_{p\ prime} U_p(v_p(E), F)$, define the function $f(c_E, A_F)$ on $\mathbb{H}$ by:

$$f(c_E, A_F) = \sum_{n \geq 1} a_n(c_E, A_F) q^n,$$

where

$$a_n(c_E, A_F) = A_F(n^{sf}) n^{\frac{k-2}{4}} \prod_p c_{E,p}(n).$$

Denote by $N(F)$ the integer $\prod_p p^{n_p}$, and by $U(E, F, A_F)$ the complex vector space generated by the set $\{f(c_E, A_F)\}_{c_E \in U(E,F)}$.

We apply the theorem of Waldspurger to compute a basis for the space $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ where

$$F = \sum_{n \geq 1} b_n q^n \in S_{k-1}^{new}(\Gamma_0(N))$$

with odd square-free level N, $k \equiv 3 \mod 4$ and $L(F, \frac{k-1}{2}) \neq 0$.

Since F has an odd square-free level and χ_0 is the trivial character modulo $4N$, we get

$$n_p = \begin{cases} 2 & \text{if } p = 2 \\ 1 & \text{if } p|N \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $N(F) = \prod p^{n_p} = 4N$. Also,

$$S_{\frac{k}{2}}(\Gamma_0(4N), F) = U(4N, F, A_F).$$

Our goal is thus reduced to computing the space $U(4N, F, A_F)$.

For every $p \nmid N$ set

$$\lambda_p = b_p p^{1-\frac{k}{2}}, \quad \alpha_p + \alpha'_p = \lambda_p \text{ and } \alpha_p \alpha'_p = 1$$

For every $p|N$ set

$$\lambda'_p = b_p p^{1-\frac{k}{2}}$$

We get,

$$U_p(v_p(4N), F) = \begin{cases} \{c_p^0[\lambda_p]\} & \text{if } p \nmid 4N \\ \{c_p^s[\lambda'_p]\} & \text{if } p|N \\ \{c'_2[\alpha_2], c'_2[\alpha'_2]\} & \text{if } p = 2 \text{ and } \alpha_2 \neq \alpha'_2 \\ \{c'_2[\alpha_2], c''_2[\alpha_2]\} & \text{otherwise} \end{cases}$$

The set $U(4N, F)$ consists of

$$\mathbf{c_1} = c'_2[\alpha_2] \prod_{p|N} c_p^s[\lambda'_p] \prod_{\substack{p \neq 2 \\ p \nmid N}} c_p^0[\lambda_p] \text{ and } \mathbf{c_2} = c'_2[\alpha'_2] \prod_{p|N} c_p^s[\lambda'_p] \prod_{\substack{p \neq 2 \\ p \nmid N}} c_p^0[\lambda_p].$$

Hence, a basis for $S_{\frac{k}{2}}(\Gamma_0(4N), F) = U(4N, F, A_F)$ consists of the functions

$$f(\mathbf{c_1}, A_F) = \sum_{n \geq 1} a_n(\mathbf{c_1}, A_F) q^n \text{ and } f(\mathbf{c_2}, A_F) = \sum_{n \geq 1} a_n(\mathbf{c_2}, A_F) q^n$$

with

$$a_n(\mathbf{c_1}, A_F) = A_F(n^{sf}) n^{\frac{k-2}{4}} c_2'[\alpha_2](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{\substack{p \neq 2 \\ p \nmid N}} c_p^0[\lambda_p](n)$$

and

$$a_n(\mathbf{c_2}, A_F) = A_F(n^{sf}) n^{\frac{k-2}{4}} c_2'[\alpha_2'](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{\substack{p \neq 2 \\ p \nmid N}} c_p^0[\lambda_p](n).$$

Proposition

We have

$$a_n(\mathbf{c_1}, A_F) = a_n(\mathbf{c_2}, A_F),$$

for every positive integer n *such that* $(-1)^{\lambda} n = -n \equiv 2, 3 \mod 4$. *Therefore, the form* $f(\mathbf{c_1}, A_F) - f(\mathbf{c_2}, A_F)$ *belongs to the Kohnen subspace.*

Corollary

The half-integral weight modular form g is a non-zero scalar multiple of $f(\mathbf{c_1}, A_F) - f(\mathbf{c_2}, A_F)$.

Theorem (H.)

For a positive square-free integer t satisfying $\left(\frac{\Delta_{-t}}{p}\right) = -p^{\frac{1}{2}}\lambda'_p$ for all p such that $p|N$ and $p \nmid \Delta_{-t}$, we have

$$A_F(t) = r2^{\frac{\nu}{2}}|\Delta_{-t}|^{\frac{2-k}{4}} c_{|\Delta_{-t}|},$$

where r is a complex constant depending only on F, and $2^{\frac{\nu}{2}} = \prod_{\substack{p|N \\ p\nmid\Delta_{-t}}} 2^{-\frac{1}{2}}$.

Remark

Let n be a positive integer such that its square-free part which we denote by n' satisfies $\left(\frac{\Delta_{-n'}}{p}\right) = p^{\frac{1}{2}}\lambda'_p$ for some prime p such that $p|N$ and $p \nmid \Delta_{-n'}$. For all such n, we have

$$a_n(\mathbf{c_1}, A_F) = a_n(\mathbf{c_2}, A_F) = 0$$

regardless of the value of $A_F(n')$.

Define $K_1(n)$ and $K_2(n)$ by:

$$K_1(n) = c'_2[\alpha_2](n) \prod_{p|N} c^s_p[\lambda'_p](n) \prod_{\substack{p\neq 2 \\ p\nmid N}} c^0_p[\lambda_p](n)$$

and

$$K_2(n) = c'_2[\alpha'_2](n) \prod_{p|N} c^s_p[\lambda'_p](n) \prod_{\substack{p\neq 2 \\ p\nmid N}} c^0_p[\lambda_p](n)$$

Theorem (H.)

Let F be a newform in $S_{k-1}^{new}(\Gamma_0(N))$ with odd square-free level N such that $k \equiv 3 \mod 4$ and $L(F, \frac{k-1}{2}) \neq 0$. The space $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ is generated by $f(\mathbf{c_1}, A_F) = \sum_{n\geq 1} a_n(\mathbf{c_1}, A_F)q^n$ and $f(\mathbf{c_2}, A_F) = \sum_{n\geq 1} a_n(\mathbf{c_2}, A_F)q^n$ where

$$a_n(\mathbf{c_1}, A_F) = \begin{cases} 2^{\frac{\nu}{2}} c_{|\Delta_{-n'}|} |\Delta_{-n'}|^{\frac{2-k}{4}} n^{\frac{k-2}{4}} K_1(n) & \text{if } \left(\frac{\Delta_{-n'}}{p}\right) = -p^{\frac{1}{2}} \lambda'_p \ \forall p|N\text{: } p \nmid \Delta_{-n'} \\ 0 & \text{otherwise} \end{cases}$$

and

$$a_n(\mathbf{c_2}, A_F) = \begin{cases} 2^{\frac{\nu}{2}} c_{|\Delta_{-n'}|} |\Delta_{-n'}|^{\frac{2-k}{4}} n^{\frac{k-2}{4}} K_2(n) & \text{if } \left(\frac{\Delta_{-n'}}{p}\right) = -p^{\frac{1}{2}} \lambda'_p \ \forall p|N\text{: } p \nmid \Delta_{-n'} \\ 0 & \text{otherwise} \end{cases}$$

The terms $c_{|\Delta_{-n'}|}$ are determined by the Fourier expansion $g = \sum_{n\geq 1} c_n q^n$.

Example

Let F be the newform in $S_2^{new}(\Gamma_0(15))$ corresponding to the elliptic curve

$$y^2 + xy + y = x^3 + x^2 - 10x - 10.$$

We have

$$F = \sum_{n \geq 1} b_n q^n = q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 - q^{10} - 4q^{11} + O(q^{12}).$$

The space $S_{\frac{3}{2}}(\Gamma_0(60), F)$ is generated by the forms g and h, where

$$g = q^3 - 2q^8 - q^{15} + 2q^{20} + 2q^{23} + O(q^{24})$$

and

$$h = -4q^2 + q^3 + 4q^5 + 2q^8 - 4q^{12} + 3q^{15} + 4q^{18} - 2q^{20} - 6q^{23} + O(q^{24})$$