Skinner

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PS1

p-adic L-functions, Iwasawa theory, and elliptic curves

$E/\mathbb{Q}$ elliptic curve

$$\left.\begin{array}{l} \underset{\substack{\parallel \\ r}}{\mathrm{rk}\ E(\mathbb{Q})} \overset{?}{=} \mathrm{ord}_{s=1} L(E,s) \\[2em] \dfrac{L^{(r)}(E,1)}{r!} = \Omega_{E/\mathbb{Q}} \dfrac{\#Ш(E/\mathbb{Q})}{(\#E(\mathbb{Q})_{tors})^2} R_{E/\mathbb{Q}} \prod_{\ell} c_\ell(E/\mathbb{Q}) \end{array}\right] \text{BSD}$$

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) \to Ш_{p^\infty}(E/\mathbb{Q}) \to 0$$

GZ-K:

$$\mathrm{ord}_{s=1} L(E,s) = r \le 1 \;\overset{?}{\Leftarrow}\; \underset{GZ}{\Rightarrow} \;\; \begin{array}{l} \mathrm{rk}\ E(\mathbb{Q}) = r \\ \#Ш(E/\mathbb{Q}) < \infty \end{array}$$

$$\text{Euler systems } \Downarrow \;\; ?\Uparrow \qquad \Uparrow \forall p \;\; \Downarrow$$

$$\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) \text{ has } \mathbb{Z}_p\text{-corank} = r$$

Thm A: Let $E/\mathbb{Q}$ be semistable, $p \ge 5$ a prime of good reduction. Suppose

(a) $E[p]$ is irreducible

(b) if $N_E = 2\ell$, then $p \nmid c_2(E/\mathbb{Q})$

(c) if $\ell \mid N_E$, $\ell \equiv \pm 1 \pmod p$, then $p \nmid c_\ell(E/\mathbb{Q})$

If the corank of $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ is 1, then $\mathrm{rk}\ E(\mathbb{Q}) = 1$, $\mathrm{ord}_{s=1} L(E,s) = 1$, and $\#Ш(E/\mathbb{Q}) < \infty$.

Thm B: For E and p as in Theorem A, satisfying (a) and (b), if $\mathrm{ord}_{s=1} L(E,s) = r \le 1$, then

$$\mathrm{ord}_p\left(\frac{L^{(r)}(E,1)}{\Omega_{E/\mathbb{Q}}\, R_{E/\mathbb{Q}}}\right) = \mathrm{ord}_p\left(\#Ш(E/\mathbb{Q}) \prod_{\ell} c_\ell\right).$$

<u>Thm A</u>: S-U, W. Zhang, X. Wan

<u>Thm B</u>: S-U, W. Zhang, X. Wan, D. Jetchev.

Let $V = T_pE \otimes_{\mathbb{Z}} \mathbb{Q}_p$ and $H^1_f(\mathbb{Q}, V)$ be the Bloch-Kato cohomology group. We have

$$\dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}, V) = \mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}).$$

Assume $H^1_f(\mathbb{Q}, V)$ has dim. 1.

- parity conjecture $\Rightarrow \varepsilon(E) = -1$
- choose $K/\mathbb{Q}$ imag. quad. field s.t. $\dim H^1_f(K, V) = 1$
  ($L(E_K, 1) \neq 0$).

Expect to prove that the Heegner pt. $P_K \in E(K)$ is nontorsion.

- p splits in K, $p = v\bar{v}$.

  $L_v(E/K, \Psi)$ ... interpolates special values

  $L(V^*(1) \otimes \Psi, 0)$

  anticyclotomic char. H.T. wts of Ψ $\underset{v\ \ \bar{v}}{(-n, n)}$. , $n > 0$.

$$\begin{array}{c} K_\infty \\ \Gamma \Big| \ {}^{a.c.}_{\mathbb{Z}_p\text{-ext}} \\ K \end{array}$$

Bertolini-Darmon-Prasanna:

$$L_v(E/K, \mathbb{1}) = (*)\left(\log_{E(K_v)} P_K\right)^2$$

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Iwasawa theory suggests

$$L_v(E/K, \mathbb{1}) \neq 0 \iff H^1_v(K, V) = 0.$$

$$(\mathrm{res}_w c = 0 \quad \forall w \neq \bar{v})$$

X. Wan proved enough of the main conjecture for $L_v(E/K, \psi)$ to conclude "$\Leftarrow$" holds.

$$\left(L_v(E/K, \psi) \mid \mathrm{char}_{\mathbb{Z}_p[[\Gamma]]} \left(\begin{matrix}\text{Greenberg} \\ \text{Selmer grp}\end{matrix} \right) \right)$$

$\otimes \mathbb{Q}_p$ ← can be removed via Bungle (student of Hida)

Assume

$$\dim_{\mathbb{Q}} H^1_f(K, V) = 1$$

and

$$\underbrace{H^1_f(K, V) \hookrightarrow H^1_f(K_v, V)}_{\text{would like to remove this assumption}}$$

duality $\Rightarrow H^1_v(K, V) = 0$

$$\Rightarrow L_v(E/K, \mathbb{1}) \neq 0$$

$$\Rightarrow P_K \text{ is non-torsion.}$$

Can be removed via arguments of W. Zhang.

From the divisibility of the char. ideal

$$\#Ш_{p^\infty}(E/\mathbb{Q}) \, [E(K_v) : \widehat{E(K)}]^2$$

$$\mathbb{Z}_p / (\log_{E(K_v)} P_K)^2 \leq \# \mathrm{Sel}_v(E/K) \cdot \underbrace{\prod_{\substack{l \mid N_E \\ \text{split in } K}} c_l(E/\mathbb{Q}_l)^2}$$

$$= [E(K_v) : \mathbb{Z}_p P_K]^2$$

(2 split in K)

in Thm A choose K so this is a unit. in Thm B choose K so at most one such l.

$$\Rightarrow \quad [E(K):\mathbb{Z}P_K]_{p^n} \leq \#Ш(E/\mathbb{Q}) \prod_{\ell} c_\ell(E/\mathbb{Q}_\ell)^2$$

Now one uses GZ:

$$\frac{L'(E/K,1)}{\Omega_{E/K}} = [E(K):\mathbb{Z}P_K]^2 R_{E/K} \cdot \frac{\langle f_E, f_E\rangle}{\langle f_E', f_E'\rangle} \times \prod_{\ell \text{ split}} c_\ell^2 \times \prod_{\ell \text{ inert}}'' c_\ell .$$

$$\Rightarrow \quad \mathrm{ord}_p(\quad) \leq \mathrm{ord}_p\Big(\#Ш(E/K) \times \prod_{\ell \text{ split}} c_\ell^2 \prod_{\ell \text{ inert}} c_\ell\Big)$$

Use work of a student from Berkeley to get equality (result about tamagawa factors)

Still need to separate $L(E_K,1)$ out of the formula.

If E is ordinary at p, use the special value formula for

$$\frac{L(E_K,1)}{\Omega_{E_K}}$$ coming from the MC for E_K.

If E is supersingular at p, combine with work of Kobayashi to deduce the supersingular MC for E and E_K to get the desired result for $L(E_K,1)$.