

Finite dimensional Banach Spaces:

p prime number, $|\cdot|_p$ on \mathbb{Q} $|p|_p = p^{-1}$, $|xy|_p = |x|_p |y|_p$,

$$|x+y|_p \leq \max(|x|_p, |y|_p)$$

$$\mathbb{Q}_p = \widehat{\mathbb{Q}} \text{ w.r.t. } |\cdot|_p.$$

$$\mathbb{Q}_p \supseteq \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

$$= \varprojlim \mathbb{Z}/p^n \mathbb{Z}$$

$$\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$$

$\overline{\mathbb{Q}_p}$ = alg. closure of \mathbb{Q}_p . ($[\overline{\mathbb{Q}_p} : \mathbb{Q}_p] = \infty$ e.g. $X^n - p$ is irred. in $\mathbb{Q}_p[X]$ for all n)

$|\cdot|_p$ extends uniquely to $\overline{\mathbb{Q}_p}$. $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts via isometries.

$$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}} \supseteq G_{\mathbb{Q}_p} \text{ (extends via continuity)}$$

"Aut_{cts}(\mathbb{C}_p).

~~$\mathbb{C}_p \cong \mathbb{C}$~~ . (uses axiom of choice so forget this isom.)

$[\mathbb{C}_p : \mathbb{Q}_p]$ not countable.

Tate (1966): $2\pi i \notin \mathbb{C}_p$.

- Hint: $e^{2\pi i/p^n}$ compute: $\log e^{2\pi i/p^n} = 0$

$$\log x = \sum \frac{(-1)^{n-1}}{n} (x-1)^n.$$

$$B(1, 1) \xrightarrow{\log} \mathbb{C}_p \rightarrow 0 \text{ is exact.}$$

↑

$M_{p^{\infty}}$

↑

0

- Want $\sigma(e^{2\pi i/p^n}) = e^{\chi(\sigma) 2\pi i/p^n}$

$\chi: G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times$ cyclotomic char.

Want $\sigma(2\pi i) = \chi(\sigma) 2\pi i \stackrel{\text{Toti}}{\Rightarrow} 2\pi i = 0.$

Fontaine (~1980) constructed a natural ring $B_{dR}^+ \ni 2\pi i = t$

$\sigma(t) = \chi(\sigma)t$

$\Theta: B_{dR}^+ \rightarrow \mathbb{C}_p$ kernel generated by $t.$

$\mathbb{C}_p = B_{dR}^+ / t B_{dR}^+.$

$B_{dR}^+ \cong \widehat{\mathbb{C}_p[[t]]}$ again uses axiom of choice.
 $\overline{\mathbb{Q}_p}$ is dense in $B_{dR}^+.$

$B_{dR}^+ / t^n B_{dR}^+$ is the completion of $\overline{\mathbb{Q}_p}$ for $| \cdot |_{p,1}$.

$x \in \overline{\mathbb{Q}_p}, x = Q(\pi), Q \in \mathbb{Q}_p[x]$

π is killed by an Eisenstein polynomial P

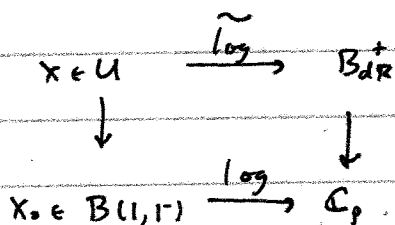
$\frac{dx}{d\pi} = \frac{-Q'(\pi)}{P'(\pi)}$

$|x|_{p,1} = \max(|x|_p, |\frac{dx}{d\pi}|_p).$

$0 \rightarrow t B_{dR}^+ / t^2 B_{dR}^+ \rightarrow B_{dR}^+ / t^2 B_{dR}^+ \rightarrow B_{dR}^+ / t B_{dR}^+ \rightarrow 0$ ($B_{dR}^+ / t \cong \mathbb{C}_p$)

" \mathbb{C}_p " \mathbb{C}_p

$U = \{ (x_0, x_1, \dots, x_n, \dots) : x_{n+1}^p = x_n, x_n \in B(1,1) \}$



$$\tilde{\log}(1, \dots, e^{2\pi i/p^n}) = \mathbb{Z}$$

$$0 \rightarrow \mathbb{Q}_p \mathbb{Z} \rightarrow U \rightarrow \mathbb{C}_p \rightarrow 0$$

$$U \sim \mathbb{C}_p \oplus \mathbb{Q}_p$$

$$B_{\text{cris}}^+ \subset B_{\text{dR}}^+ \quad B_{\text{cris}}^+ \ni \text{Fr.} \quad x \mapsto x^p$$

$$\tilde{\log} : U \xrightarrow{\sim} (B_{\text{cris}}^+)^{q=p} \quad (\log x^p = p \log x)$$

$$0 \rightarrow \mathbb{Q}_p \mathbb{Z}^m \rightarrow (B_{\text{cris}}^+)^{q=p^m} \rightarrow B_{\text{dR}}^+ / \mathbb{Z}^m \rightarrow 0$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathbb{C}_p^m \oplus \mathbb{Q}_p & & \mathbb{C}_p^m \end{array}$$

Problem: $\mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{Q}_p$, so how does one keep track of the \mathbb{Q}_p -structure?

Finite dimensional Vector Spaces (~ 2000 , Fontaine - Illus, Fargue, Scholze)

a Banach \mathbb{Q}_p -algebra ($\|xy\| \leq \|x\| \|y\|$, $\|x+y\| \leq \max(\|x\|, \|y\|)$)

is nice if

$$\bullet \|x\| = \max_{\lambda: \Lambda \rightarrow \mathbb{C}_p} |\lambda(x)|_p$$

$\bullet x \mapsto x^p$ is surjective.

$\bullet \Lambda = \mathbb{C}_p$ is nice.

$\bullet \Lambda \rightarrow \tilde{\Lambda}$

A v.s. is a functor from nice algebras to \mathbb{Q}_p -vector spaces

Examples:

1) V a f.d. \mathbb{Q}_p -v.s. This can be viewed as a V.S.

$$\text{via } V(\Lambda) = V \quad \forall \Lambda,$$

$$V(\Lambda_1) \xrightarrow{\text{id}} V(\Lambda_2).$$

2) $d \in \mathbb{Z}$, W^d : $V^d(\Lambda) = \Lambda^d$

$$\Lambda_1 \rightarrow \Lambda_2 \rightsquigarrow \Lambda_1^d \rightarrow \Lambda_2^d.$$

A. V.S. W is finite dimensional:

$$\begin{array}{ccccccc} 0 & \rightarrow & V_2 & & & & \\ & & \searrow & & & & \\ 0 & \rightarrow & V_2 & \rightarrow & W' & \rightarrow & W^d \rightarrow 0 \quad (*) \\ & & & & \searrow & & \\ & & & & & & W \rightarrow 0 \end{array}$$

$$\dim W = d$$

$$\text{ht } W = \dim_{\mathbb{Q}_p} V_1 - \dim_{\mathbb{Q}_p} V_2, \quad \text{Dim } W \text{ " } \text{Dim } (\dim W, \text{ht } W)$$

Thm: (i) $\text{Dim } W$ is well-defined

(ii) $f: W_1 \rightarrow W_2$ $\ker f$ and $\text{Im } f$ are f.d. V.S.

$$\text{Dim } W_1 = \text{Dim } \ker f + \text{Dim } \text{Im } f.$$

(iii) if $\dim W = 0 \Rightarrow \text{ht } W \geq 0$

(iv) $W \subseteq W' \Rightarrow W$ is f.d. or finite dim. over \mathbb{Q}_p

$$\Rightarrow \text{ht } W \geq 0.$$

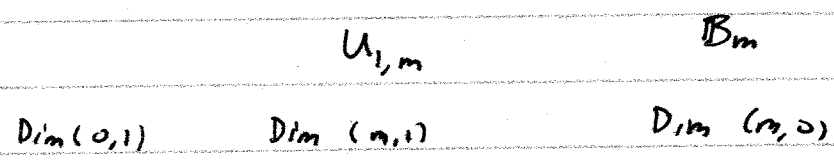
Examples:

1) $m \geq 1$ $B_m = \mathbb{B}_{\text{dR}}^+ / t^m \mathbb{B}_{\text{dR}}^+$

$$\text{Dim } B_m = (m, 0)$$

2) a, b $U_{a,b} = (\mathbb{B}_{\text{Cris}}^+)^{\varphi^a = p^b}$
 $\text{Dim } U_{a,b} = (b, a)$

$$0 \rightarrow \mathbb{Q}_p \xrightarrow{\iota^m} (\mathbb{B}_{\text{Cris}}^+)^{\varphi = p^m} \rightarrow \mathbb{B}_{\text{dR}}^+ / \iota^m \rightarrow 0$$



Comparison theorem and p -adic periods $(\int_{\mathbb{C}} \frac{dz}{z} = 2\pi i)$

X/\mathbb{Q} , projective smooth

$X(\mathbb{C})$

$$H_{\text{dR}}^i(X(\mathbb{C})) \times H_i(X(\mathbb{C}), \mathbb{Z}) \rightarrow \mathbb{C}$$

$$(\omega, u) \longmapsto \int_u \omega$$

Can take dual of this

$$H_{\mathbb{B}}^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \simeq H_{\text{dR}}^i(X(\mathbb{C}), \mathbb{C})$$

$\swarrow \mathbb{Q}$ -v.s.

Want p -adic version

$$\mathbb{Q}_p \otimes H_{\mathbb{B}}^i(X, \mathbb{Q})$$

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$$H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$$

$$\mathbb{B}_{\text{dR}}^+[\frac{1}{\iota}] \otimes H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \simeq \mathbb{B}_{\text{dR}}^+[\frac{1}{\iota}] \otimes H_{\text{dR}}^i(X)$$

$\curvearrowright G_{\mathbb{Q}_p}$, Fil

$$B_{\text{cris}}^+[\frac{1}{t}] \otimes H_{\text{ct}}^i(X_{\text{dR}}, \mathbb{Q}_p) \cong B_{\text{cris}}^+[\frac{1}{t}] \otimes H_{\text{dR}}^i(X) \quad \text{DP}$$

Lots of work ... gives ($r \gg 0$)

$$\begin{array}{c} \dots \rightarrow [H_{\text{ct}}^i \rightarrow (t^{-r} B_{\text{cris}}^+ \otimes H_{\text{dR}}^i)^{\varphi=1} \\ \downarrow \leftarrow \text{want surjectivity} \\ (t^{-r} B_{\text{dR}}^+ \otimes H_{\text{dR}}^i) / \text{Fil}^0 \leftarrow \text{want this part} \\ \downarrow \text{to be exact.} \\ H_{\text{ct}}^{i,r} \\ \vdots \end{array}$$

$$\begin{array}{ccc} H_{\text{ct}}^i & (b; a) & (b', a') \\ \text{Dim}(0, a_i) & & \text{Dim}(\text{DMSA}) \\ & & (0, a_{i+1}) \end{array}$$

$$ht(\text{Im } c) \geq 0 \quad \dim(\text{coker})$$

\vdots