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Finite dimensional Banach spaces.

p prime numbers, $|\cdot|_p$ on $\mathbb{Q}$ $|p|_p = p^{-1}$, $|xy|_p = |x|_p |y|_p$,

$|x+y|_p \leq \max(|x|_p, |y|_p)$

$\mathbb{Q}_p = \hat{\mathbb{Q}}$ wrt. $|\cdot|_p$.

$\mathbb{Q}_p \supseteq \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$

$= \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

$\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$.

$\overline{\mathbb{Q}}_p$ = alg. closure of $\mathbb{Q}_p$. ($[\overline{\mathbb{Q}}_p : \mathbb{Q}_p] = \infty$ e.g. $X^n - p$ is irred. in $\mathbb{Q}_p[X]$ for all n)

$|\cdot|_p$ extends uniquely to $\overline{\mathbb{Q}}_p$. $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts via isometries.

$\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p \circlearrowleft \underset{\overset{\|}{\mathrm{Aut}_{cts}(\mathbb{C}_p)}}{G_{\mathbb{Q}_p}}$ (extends via continuity)

$\cancel{\mathbb{C}_p \cong \mathbb{C}}$. (uses axiom of choice so forget this isom.)

$[\mathbb{C}_p : \mathbb{Q}_p]$ not countable.

Tate (1966): $2\pi i \notin \mathbb{C}_p$.

- Hint: $e^{2\pi i/p^n}$ compute: $\log e^{2\pi i/p^n} = 0$

$$\log x = \sum \frac{(-1)^{n-1}}{n}(x-1)^n.$$

$$\begin{tikzcd} B(1,1^-) \arrow[r, "\log"] & \mathbb{C}_p \arrow[r] & 0 \\ \mu_{p^\infty} \arrow[u] & & \\ 0 \arrow[u] & & \end{tikzcd}$$

is exact.

- Want $\sigma(e^{2\pi i/p^n}) = e^{\chi(\sigma) 2\pi i/p^n}$

$\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ cyclotomic char.

Want $\sigma(2\pi i) = \chi(\sigma)\, 2\pi i \overset{\text{Tate}}{\Longrightarrow} 2\pi i = 0.$

Fontaine ($\sim$1980) constructed a natural ring $B_{dR}^+ \ni 2\pi i = t$ $\circlearrowleft G_{\mathbb{Q}_p}$

$\sigma(t) = \chi(\sigma) t$

$\theta : B_{dR}^+ \twoheadrightarrow \mathbb{C}_p$ kernel generated by t.

$\mathbb{C}_p = B_{dR}^+ / t B_{dR}^+.$

$B_{dR}^+ \cancel{\simeq \mathbb{C}_p[[t]]}$ again uses axiom of choice.

$\overline{\mathbb{Q}}_p$ is dense in B_{dR}^+.

$B_{dR}^+ / t^2 B_{dR}^+$ is the completion of $\overline{\mathbb{Q}}_p$ for $|\cdot|_{p,1}$.

$x \in \overline{\mathbb{Q}}_p$, $X = Q(\pi)$, $Q \in \mathbb{Q}_p[\mu_N][X]$

π is killed by an Eisenstein polynomial P

$$\frac{dx}{dp} = \frac{-Q'(\pi)}{P'(\pi)}$$

$$|x|_{p,1} = \max\left(|x|_p, \left|\frac{dx}{dp}\right|_p\right).$$

- $$\begin{tikzcd} 0 \arrow[r] & t B_{dR}^+ / t^2 B_{dR}^+ \arrow[r] \arrow[d, phantom, "\parallel"] & B_{dR}^+ / t^2 B_{dR}^+ \arrow[r] & B_{dR}^+ / t B_{dR}^+ \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & \mathbb{C}_p t & & \mathbb{C}_p & \end{tikzcd}$$ $\left(B_{dR}^+/t^2 \simeq \mathbb{C}_p^2\right)$

- $U = \{(x_0, x_1, \ldots, x_n, \ldots) : x_{n+1}^p = x_n,\ x_n \in B(1, 1^-)\}$

$$\begin{tikzcd} x \in U \arrow[r, "\widetilde{\log}"] \arrow[d] & B_{dR}^+ \arrow[d] \\ x_0 \in B(1, 1^-) \arrow[r, "\log"] & \mathbb{C}_p \end{tikzcd}$$

$$\widetilde{\log}(1, \dots, e^{2\pi i/p^n}) = t$$

$$0 \to \mathbb{Q}_p t \to U \to \mathbb{C}_p \to 0$$

$$U \sim \mathbb{C}_p \oplus \mathbb{Q}_p$$

$B^+_{cris} \subset B^+_{dR}$ $\quad B^+_{cris} \circlearrowleft \varphi$ Frob. $\quad x \mapsto x^p$

$\widetilde{\log} : U \xrightarrow{\sim} (B^+_{cris})^{\varphi = p}$ $\quad (\log x^p = p \log x)$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q}_p t^m \arrow[r] & (B^+_{cris})^{\varphi = p^m} \arrow[r] \arrow[d, phantom, "\wr"] & B^+_{dR}/t^m \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\ & & \mathbb{C}_p^m \oplus \mathbb{Q}_p & \mathbb{C}_p^m & \end{tikzcd}$$

Problem: $\mathbb{C}_p \simeq \mathbb{C}_p \oplus \mathbb{Q}_p$, so how does one keep track of the $\mathbb{Q}_p$-structure?

Finite dimensional Vector Spaces (~2000, Fontaine-Plut, Fargues, Scholze)

a Banach $\mathbb{Q}_p$-algebra ($\|xy\| \le \|x\| \|y\|$, $\|xy\| \le \max(\|x\|, \|y\|)$)

is nice if

- $\|x\| = \max\limits_{\lambda : \Lambda \to \mathbb{C}_p} |\lambda(x)|_p$
- $x \mapsto x^p$ is surjective.

- $\Lambda = \mathbb{C}_p$ is nice.
- $\Lambda \longrightarrow \tilde{\Lambda}$

A V.S. is a functor from nice algebras to $\mathbb{Q}_p$-vector spaces

Examples:

1) V a f.d. $\mathbb{Q}_p$-v.s. This can be viewed as a V.S.

via $V(\Lambda) = V \quad \forall \Lambda$,

$V(\Lambda_1) \xrightarrow{id} V(\Lambda_2)$.

2) $d \in \mathbb{Z}$, $\mathbb{V}^d$: $\mathbb{V}^d(\Lambda) = \Lambda^d$

$\Lambda_1 \to \Lambda_2 \rightsquigarrow \Lambda_1^d \to \Lambda_2^d$.

A. V.S. $\mathbb{W}$ is finite dimensional:

$$\begin{tikzcd}
0 \arrow[r] & V_2 \arrow[dr] & & & & \\
0 \arrow[r] & V_2 \arrow[r] & \mathbb{W}' \arrow[r] \arrow[dr] & \mathbb{V}^d \arrow[r] & 0 & (*) \\
& & & \mathbb{W} \arrow[r] & 0 &
\end{tikzcd}$$

$\dim \mathbb{W} = d$

$\mathrm{ht}\,\mathbb{W} = \dim_{\mathbb{Q}_p} V_1 - \dim_{\mathbb{Q}_p} V_2$, ~~Dim~~ $\overset{\mathrm{Dim}\,\mathbb{W}}{\shortparallel}$ $(\dim \mathbb{W}, \mathrm{ht}\,\mathbb{W})$

Thm: (i) $\mathrm{Dim}\,\mathbb{W}$ is well-defined

(ii) $f : \mathbb{W}_1 \to \mathbb{W}_2$ $\ker f$ and $\mathrm{Im}\, f$ are f.d. V.S.

$\mathrm{Dim}\,\mathbb{W}_1 = \mathrm{Dim}\, \ker f + \mathrm{Dim}\, \mathrm{Im}\, f$.

(iii) if $\dim \mathbb{W} = 0 \Rightarrow \mathrm{ht}\,\mathbb{W} \geq 0$

(iv) $\mathbb{W} \subseteq \mathbb{V}^1 \Rightarrow \mathbb{W}$ is $\mathbb{V}^1$ or finite dim. over $\mathbb{Q}_p$

$\Rightarrow \mathrm{ht}\,\mathbb{W} \geq 0$.

Examples:

1) $m \geq 1$ $\mathbb{B}_m = \mathbb{B}_{dR}^+ / t^m \mathbb{B}_{dR}^+$

$\mathrm{Dim}\, \mathbb{B}_m = (m, 0)$

2) a, b $\quad U_{a,b} = (\mathbb{B}_{cris}^{+})^{\varphi^a = p^b}$

$$\mathrm{Dim}\, U_{a,b} = (b, a)$$

$$0 \to \mathbb{Q}_p t^m \to (\mathbb{B}_{cris}^{+})^{\varphi = p^m} \to \mathbb{B}_{dR}^{+}/t^m \to 0$$

$$U_{1,m} \qquad\qquad \mathbb{B}_m$$

$$\mathrm{Dim}(0,1) \qquad \mathrm{Dim}\ (m,1) \qquad \mathrm{Dim}\ (m,0)$$

Comparison theorems and p-adic periods $\quad \left(\int_C \frac{dz}{z} = 2\pi i \right)$

$X/\mathbb{Q}$, projective smooth

$X(\mathbb{C})$

$$H^i_{dR}(X(\mathbb{C})) \times H_i(X(\mathbb{C}), \mathbb{Z}) \longrightarrow \mathbb{C}$$

$$(\omega, u) \longmapsto \int_u \omega.$$

Can take dual of this

$$H^i_B(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \simeq \underbrace{H^i_{dR}(X/\mathbb{Q})}_{\mathbb{Q}\text{-v.s.}} \otimes \mathbb{C}$$

Want p-adic version

$$\mathbb{Q}_p \otimes H^i_B(X, \mathbb{Q})$$

$$\shortparallel$$

$$H^i_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$$

$$B_{dR}^{+}[\tfrac{1}{t}] \otimes H^i_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq B_{dR}^{+}[\tfrac{1}{t}] \otimes H^i_{dR}(X) \circlearrowleft G_{\mathbb{Q}_p},\ \mathrm{Fil}$$

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$$B^+_{cris}[\tfrac{1}{t}] \otimes H^i_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq B^+_{cris}[\tfrac{1}{t}] \otimes H^i_{dR}(X) \quad D\varphi$$

Lots of work .... gives $(r \gg 0)$

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_{et} \arrow[r] & (t^{-r} B^+_{cris} \otimes H^i_{dR})^{\varphi=1} \arrow[d, "c"'] \\
& & (t^{-r} B^+_{dR} \otimes H^i_{dR})/\mathrm{Fil}^0 \arrow[d] \\
& & H^{i+1}_{et} \arrow[d] \\
& & \vdots
\end{tikzcd}$$

want surjective.

want this part to be exact.

H^i_{et} $\quad (b_i, a_i) \quad (b_i', \overset{0}{\cancel{a_i}}) \quad H^{i+1}_{et}$

Dim $(0, a_i)$ $\qquad$ Dim ~~[illegible]~~ $(0, a_{i+1})$

$ht(\mathrm{Im}\, c) \geq 0 \quad$ dim (coker

$\vdots$