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Kolyvagin's Approach to the Birch and Swinnerton-Dyer Conjecture:

§1 BSD:

$E/\mathbb{Q}$ an elliptic curve

Conj (BSD): $\underset{r_{alg}}{\operatorname{rank} E(\mathbb{Q})} = \underset{r_{an}}{\operatorname{ord}_{s=1} L(E,s)}$

Conj (Shaf-Tate) (ST): Ш$(E/\mathbb{Q})$ is finite

These are both theorems when $r_{an} \leq 1$, mainly due to Kolyvagin.

Assume now that $r_{an} \geq 1$.

1. ✓ E is modular (Wiles, Taylor, et.al) $X_0(N) \twoheadrightarrow E$
2. ✓ Choose a quad. imag field K s.t.
   - $(D_K, 2N) = 1$
   - $D_K \equiv \square \pmod{4N}$
   - $r_{E^K, an} = \begin{cases} 0 & r_{an} \text{ odd} \\ 1 & r_{an} \text{ even} \end{cases}$

   (Bump, Waldspurger, ...)
3. Define a subgroup $V = V_K \subseteq E(\mathbb{Q})$ (Kolyvagin)
4. ✓ Prove that $\operatorname{rank} V = r_{an}$ $\Rightarrow$ BSD & ST are true (Kolyvagin)
5. Prove that $\operatorname{rank} V = r_{an}$ (Gross Zagier when $r_{an} = 1$)

§2 Definition of $V \subseteq E(\mathbb{Q})$:

$I_K \subseteq \mathcal{O}_K$ (ideal) s.t. $\mathcal{O}_K / I_K \cong \mathbb{Z}/N\mathbb{Z}$ (by hypothesis there exists such an ideal)

For each $\lambda \in \mathbb{N}$, let $\mathcal{O}_\lambda = \mathbb{Z} + \lambda \mathcal{O}_K$, $\gcd(\lambda, D_K N) = 1$: (λ sq. free)

$$z_\lambda = \left[\mathbb{C}/\mathcal{O}_\lambda \longrightarrow \mathbb{C}/(I_K^{-1} \cap \mathcal{O}_\lambda) \right] \in X_0(N)(K_\lambda)$$

ring class field of cond. λ.

$$\downarrow$$

$$y_\lambda = \pi(z_\lambda) \in E(K_\lambda)$$

$$P_\lambda = J_\lambda I_\lambda y_\lambda, \quad J_\lambda, I_\lambda \in \mathbb{Z}[G(K_\lambda/K)]$$

$$G(K_\lambda/K_1) \cong \prod_{p\mid\lambda} G(K_p/K_1) = \prod_{p\mid\lambda} \langle t_p \rangle$$ ← cyclic, order $p+1$.

Assume p inert in K.

$$I_\lambda = \prod_{p\mid\lambda} I_p, \quad I_p = \sum_{j=1}^{p} j\,t_p^j \in \mathbb{Z}[G(K_\lambda/K)]$$

$$J_\lambda := \sum_{g \in G(K_\lambda/K)/G(K_\lambda/K_1)} \bar{g} \in \mathbb{Z}[G(K_\lambda/K)]$$

Fix a prime ℓ. Let

$$\Lambda_n^t = \{ \lambda = p_1 \cdots p_t : \operatorname{ord}_\ell(\gcd(a_{p_i}, p_i+1)) \geq n \text{ for } i=1,\dots,t \}$$

$a_{p_i} \equiv p_i+1 \equiv 0 \pmod{\ell^n}$.

Prop: $[P_\lambda] \in \left(E(K_\lambda)/\ell^n E(K_\lambda) \right)^{G(K_\lambda/K)}$.

$$\varphi_{\lambda,\ell} : E(\mathbb{Q}) \longrightarrow \left(E(K_\lambda)/\ell^n E(K_\lambda) \right)^{Gal(K_\lambda/K)}.$$

Let

$$V_{\ell^n} = \langle \varphi_{\lambda,\ell}^{-1} (\mathbb{Z}[P_\lambda]) : \lambda \in \Lambda_n^{r_{an}-1} \rangle$$

Observation: ① $\ell^n E(\mathbb{Q}) \subseteq V_{\ell^n} \subseteq E(\mathbb{Q})$

② $V_{\ell^n} \supseteq V_{\ell^{n+1}} \supseteq \cdots \supseteq V_{\ell^\infty} = \bigcap_{n\geq 1} V_{\ell^n}$.

Define: $V = \bigcap_{\text{all } \ell} V_{\ell^\infty}$.

Conj (Pessimistic): $\operatorname{rank} V = 0 \iff r_{an} > 1$.

Conj. (optimistic): $\operatorname{rank} V = r_{an}$.

Question: Which, if either, is true?

Example: ① $E : 389a$ rank 2

V_3 $l = 3$

P_λ with $\lambda = 5$

$V_3 \not\ni 3E(\mathbb{Q})$, $E(\mathbb{Q})/3E(\mathbb{Q}) \cong (\mathbb{Z}/3\mathbb{Z})^2$

② $E : 571b :$ $l = 3$ $\lambda = 11$ $r_{an} = 2$

$F : 571 \cdot 11$ $r_{an} = 1$

Believe one of the pts of $E(\mathbb{Q})$ comes from $F(\mathbb{Q})$.

§3 A Naive Question:

$$r_{an} = 1 \Rightarrow \operatorname{rank} V = r_{an}$$

$\sigma \in Gal(K_1/K)$ $\cong$

Hilbert class field

$c = (x_1) - (\infty) \in J_0(N)(K_1)$ $A \in Cl(\mathcal{O}_{K})$

$$f_A = \sum_{n=1}^{\infty} \langle c, T_n c^\sigma \rangle q^n.$$

Thm: $f_A \in S_2(\Gamma_0(N))$ and

convolution, $r_A(n)$, $a_n(g)$

$$(f_A, g) = \frac{\sqrt{|D_K|}}{8\pi^2} L'_A(g, 1)$$

for each newform $g \in S_2^{new}(\Gamma_0(N))$.

$C^t \in J_0(N)(K_1)$ (Fix any $t \geq 0$.)

$z_\lambda \in X_0(N)(K_\lambda)$ $(\lambda \in \Lambda_n^t)$

$$Q_\lambda = I_\lambda \cdot ((z_\lambda) - (\infty)) \in \left(J_0(N)(K_\lambda)/\ell^n\right)^{G(K_\lambda/K_1)}$$

$$\varphi_{\lambda,n} : J_0(N)(K_1) \longrightarrow \left(J_0(N)(K_\lambda)/\ell^n\right)^{G(K_\lambda/K_1)}$$

$$C^t_{\ell^n} = \langle \varphi_{\lambda,n}^{-1}(\mathbb{Z}[Q_\lambda]) : \lambda \in \Lambda^t_n \rangle$$

$$C^t_{\ell^\infty} = \bigcap_n C_{\ell^n}, \quad C^t = \bigcap_\ell C^t_{\ell^\infty} \leq J_0(N)(K_1).$$

Fix a basis $c_1, \dots, c_m$ for C^t, and consider

$$f^t_A = \sum_{n=1}^\infty \det(\langle c_i, T_n c_j^\sigma \rangle) q^n.$$

<u>Questions</u>:

① do $f^t_A \in S_2(\Gamma_0(N))$?

② $(f^t_*, g) \longleftrightarrow$ "$L^{(k)}_*(g,1)$" $\quad$ g newform