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Modular Symbols for Global Fields:

$K = \mathbb{Q}$ or $\mathbb{F}_q(T)$.
A = ring of integers
$N \in A - \{0\}$
$\Gamma_1(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A) : \ N \mid c$ and $d \equiv 1 \pmod N \}$.

$$\mathcal{M}(N) = \{ \{\alpha,\beta\} : \alpha,\beta \in \mathbb{P}^1(K), \ \{\alpha,\beta\} + \{\beta,\gamma\} = \{\alpha,\gamma\}, \ \{g\alpha, g\beta\} = \{\alpha,\beta\}, \ g \in \Gamma_1(N) \}$$
$$= H^0(\Gamma_1(N), \mathrm{Div}^0(\mathbb{P}^1(K))).$$

We really just consider $K = \mathbb{F}_q(T)$ here.

Let E be an elliptic curve over K with split multiplicative reduction at ∞, conductor $N\infty$. Let K_N/K be the Ray class field of conductor N, ∞ is split in K_N.
$$E(K_N) \circlearrowleft Gal(K_N/K).$$

Thm: There is a $\chi \in \mathbb{X}_N = \mathrm{Hom}(Gal(K_N/K), \mathbb{C}^\times)$ s.t.
$$E(K_N)^\chi = 0.$$

Let K_∞ be the completion at ∞ of K, A_∞ the ring of integers. Tree $\mathcal{T}$: vertices $= PGL_2(K_\infty)/PGL_2(A_\infty)$.
edges $= PGL_2(K_\infty)/I = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL_2(A_\infty) : |c|_\infty < 1 \}$.

An end of $\mathcal{T}$ is an infinite sequence of consecutive edges (without going backwards)
$$\{\text{ends}\} \simeq \mathbb{P}^1(K_\infty) \supset \mathbb{P}^1(K).$$

$\mathcal{C}(N) = \{ f : \{\text{edges}\} \longrightarrow \mathbb{C} \ : \ f(e) = -f(e^*)$ (alternating),

$\sum_{e \to v} f(e) = 0$ (harmonic), $\Gamma_1(N)$ invariant

(modularity), f is eventually 0 on any

rational end $\}$.

$\longleftrightarrow$ automorphic forms on $GL_2(\mathbb{F}_q(t))$ special at ∞.

<u>Teitelbaum:</u> There is a pairing

$$\mathcal{M}(N) \times \mathcal{C}(N) \longrightarrow \mathbb{C}$$

$$\{\alpha, \beta\} \times f \longmapsto \int_\alpha^\beta f,$$

which is perfect on $(\mathcal{M}^0(N) \otimes \mathbb{C}) \times \mathcal{C}(N)$ where $\mathcal{M}^0(N)$ is the kernel of the map

$$\mathcal{M}(N) \longrightarrow \mathrm{Div}(\Gamma_1(N) \backslash \mathbb{P}^1(K))$$

$$\{\alpha, \beta\} \longmapsto (\Gamma_1(N)\alpha) - (\Gamma_1(N)\beta).$$

<u>Remarks:</u> • $\{g0, g\infty\}$ for $g \in GL_2(A)$ depends only on $\Gamma_1(N) g$.

• There is a bijection

$$\Gamma_1(N) \backslash GL_2(A) \simeq \text{elts of order } N \text{ of } (A/N)^2.$$

There is

$$\xi : \mathbb{C}[(A/N)^2] \longrightarrow \mathcal{M}(N),$$

$\xi(u,v) = 0$ if (u,v) is not of order N.

<u>Teitelbaum:</u> ① ξ is surjective

② The kernel of ξ is generated by

- $[u,v]$ $\quad (u,v)$ not of order N
- $[u,v] - [au, bv]$ $\quad (a, b \in \mathbb{F}_q^\times)$
- $[u,v] + [-v, u]$
- $[u,v] + [-v-u, u] + [v, -u-v]$ $\quad (u, v \in (A/N)^2)$.

C. Armana: Let $m \in A - \{0\}$, m a monic polynomial.

$$T_m \, \xi(u,v) = \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2(A) \\ a,d \text{ monic} \\ ad-bc=N \\ \deg a > \deg b \\ \deg d > \deg c}} \xi(au+cv,\, bu+dv).$$

(action of Hecke operators)

f newform:

$$\xi_f : (A/N)^2 \longrightarrow \mathbb{C}$$
$$(u,v) \longmapsto \int_{\xi(u,v)} f.$$

ξ_f determines f.

$$A/N \simeq \bigcup_{\substack{D \mid N \\ \text{monic poly}}} (A/D)^{\times} \quad : w \longmapsto \frac{wN_w'}{N} \bmod N_w' \quad \text{where } N_w' \text{ order of } w \text{ in } A/N.$$

$(u,v) \in (A/N)^2$ of order N

N = order of $u\,v$ in $(A/N)^2$

Σ_N = support of N.

$\Sigma_N = S \amalg \bar{S}$ disjoint

S = contains the support of u disjoint from the support of v.

$$\xi_f(u,v) = \sum_{\substack{\alpha,\beta \\ \text{char mod } N}} c_{\alpha,\beta}(f)\, \alpha\left(\frac{N_{\bar{S}}\, v}{N_{\bar{S}}}\right) \beta\left(\frac{N_S\, u}{N_S}\right).$$

There is a very complicated formula for $c_{\alpha,\beta}$, but it was too long to write down here!

Corl: f is determined by

① its central character

② The Euler factors at places dividing N of $L(f\otimes\chi, s)$ for $\mathrm{cond}(\chi) \mid N$

③ The local constant of functional equation of $L(f\otimes\chi, s)$ for $\mathrm{cond}(\chi) \mid N$.

④ The conductor of $f\otimes\chi$ for $\mathrm{cond}(\chi) \mid N$.

⑤ $L(f\otimes\chi, 1)$ for $\mathrm{cond}(\chi) \mid N$.

Corl: There exists χ of conductor dividing N s.t.

$$L(f\otimes\chi, 1) \neq 0.$$

Corl: Let E be an elliptic curve of cond. $N\infty$ with split mult. reduction at ∞. There exists χ s.t $L(E, \chi, 1) \neq 0$.

(0) Weil conjectures

$\Downarrow$

(1) $L(E, \chi, s)$ is rational and admits a F.E.

$\Downarrow$ converse thm

(2) There is an automorphic form attached to E.

$\Downarrow$ modular symbols.

(3) ξ_f determines f.

$\Downarrow$ formula

(4) $\exists$ χ of cond N s.t. $L(f, \chi, 1) \neq 0$

$\Downarrow$ BSD

(5) Thm

$\xi(u,v) + \xi(-v,u) = 0$ reflects the functional equations

$\xi(u,v) + \xi(-v,u+v) + \xi(-u-v,u) = 0$ reflects ? This is not understood.

How to recover $L(f,s)$ from ξ_f:

$$\sum_m \xi_{T_m f}(u,v)\,\left|A/m\right|^{-s} = L^{(\infty)}(f,s)\,\xi_f(u,v)$$

$$\parallel$$

$$\sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ as} \\ \text{in Manin}}} \xi(au+cv,\,bu+dv)\,\left|A/ad-bc\right|^{-s}.$$