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On ℓ-adic families of admissible representations of $GL_2(\mathbb{Q}_p)$:

This is currently work in progress. There is substantial overlap w/ indep. work of Emerton. The notation is unfortunately not the same!

Motivation: Passage from f to ρ_f "behaves well in families."

- congruences
- $R = T$ theorems
- p-adic modular forms

The representation theory side has been mostly done over fields. One would like to be able to do this in other contexts so as to be able to study families as well. The global case seems to be hopeless, but one can try for local Langlands to work in families.

$$\left\{\begin{array}{c}\text{irred. Admissible reps.}\\ \text{of } GL_n(\mathbb{Q}_p)/\mathbb{C}\end{array}\right\} \underset{\text{bij.}}{\overset{\text{L.L.}}{\longleftrightarrow}} \left\{\begin{array}{c}\text{Frob. s.s. Weil-Deligne reps}\\ WD_{\mathbb{Q}_p} \to GL_n(\mathbb{C})\end{array}\right\}$$

- Alex Paulin: families of adm. reps. over eigencurve
- Matthew Emerton: see his talk.

Starting point for this work is

$$\begin{array}{ccc}\left\{\begin{array}{c}\text{irred. adm. reps.}\\ \text{of } GL_n(\mathbb{Q}_p)/\overline{\mathbb{F}}_\ell\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}n\text{-dim Weil-Deligne}\\ \text{reps. over } \overline{\mathbb{F}}_\ell\end{array}\right\}\\ \cup\!| & & \\ \{\text{Supercuspidals}\} & \longleftrightarrow & \{\text{irred. reps}\}\end{array}$$

ℓ odd, $\ell \neq p$.

Given $\bar\rho : G_{\mathbb{Q}_p} \longrightarrow GL_2(\overline{\mathbb{F}}_\ell)$, there is a corresponding $\bar\pi$. Fix a finite length local $W(\overline{\mathbb{F}}_\ell)$-algebra A.

Def: ① An A-deformation π of $\bar\pi$ is an $A[GL_2(\mathbb{Q}_p)]$-module, free / A, admissible ($\pi = \varinjlim_{\substack{U \subseteq GL_2(\mathbb{Q}_p) \\ \text{compact open}}} \pi^U$, π^U is f.g for every such U),

and an isom. $\pi \otimes_A A/m \cong \bar\pi$.

② An A-deformation of $\bar\rho$ is $\rho : G_{\mathbb{Q}} \to GL_2(A)$ s.t. $\rho \otimes_A A/m \cong \bar\rho$.

Thm: If $\bar\rho$ irred., there is a natural bijection

$$\{A\text{-defs. } \pi \text{ of } \bar\pi\} \longleftrightarrow \{A\text{-def } \rho \text{ of } \bar\rho\}.$$

There is a natural isom.

$$R_{\bar\pi}^{univ} \longleftrightarrow R_{\bar\rho}^{univ}$$

uniquely characterized by inducing usual char. 0 L.L.C on $\overline{\mathbb{Q}}_\ell$-pts.

Pf: Compute both sides. Use explicit L.L.C. $\quad \varepsilon : G_E \to \overline{\mathbb{F}}_\ell^\times$, $E/\mathbb{Q}_p$ quadratic

Case 1) $\bar\rho$ primitive, i.e. not induced from a quadratic char. ($p=2$)

defs of $\bar\rho$ = defs of $\det \bar\rho$

defs of $\bar\pi$ = defs of central char

Case 2) $E/\mathbb{Q}_p$ quadratic (σ = conj)

$$\left\{ \text{char } \varepsilon : G_E \to \overline{\mathbb{F}}_\ell^\times \atop \varepsilon^\sigma \neq \varepsilon \right\} \Big/ \varepsilon \sim \varepsilon^\sigma \quad \xleftrightarrow{\text{LCFT}} \quad \left\{ \varepsilon : E^\times \to \overline{\mathbb{F}}_\ell^\times \atop \varepsilon^\sigma \neq \varepsilon \right\} \Big/ \sim$$

$$\downarrow$$

$$\bar\rho$$

$\updownarrow$ twist to make L-fns match

$\varepsilon \Delta_\varepsilon$

$\updownarrow$

type $J \subseteq GL_2(\mathbb{Q}_p)$, Λ f.d. rep of J

1) $J = \mathbb{Q}_p^\times GL_2(\mathbb{Z}_p)$,

$\Lambda = \Lambda' \otimes (\chi \circ \det)$

Λ' = inflated from $GL_2(\mathbb{F}_p)$

2) $J = E^\times U_n$, U_n is a pro-p-grp coming from "fundamental strata"

$\updownarrow$

$\bar{\pi} = c\text{-Ind}_J^{GL_2(\mathbb{Q}_p)} \Lambda$. ∎

What if $\bar{\rho}$ is not irred.? Take $\bar{\rho} : G_{\mathbb{Q}_p} \to GL_2(\bar{\mathbb{F}}_\ell)$, not irred.

$\bar{\rho}$ special $\longleftrightarrow$ $\bar{\pi}$ Steinberg

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Above proof still works here.

$\bar{\rho} = \chi_1 \oplus \chi_2$, $\chi_2 \notin \{\omega\chi_1, \chi_1, \omega^{-1}\chi_1\}$

ω = cyclotomic

Above proof still works here as well.

$\bar{\rho} = \chi_1 \oplus \chi_1$ $\quad$ $\bar{\pi}$ has nice universal def.

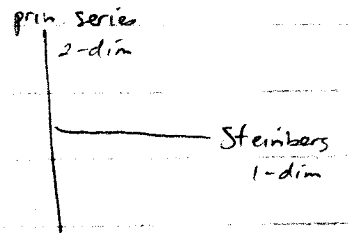
Best one can do if $p \not\equiv 1 \pmod{\ell}$ is to construct a family over $R^{ver}_{\bar{\rho}}$ lifting $\bar{\pi}$ that is "right on $\bar{\mathbb{Q}}_p$-points".

$\bar{\rho} = 1 \oplus \omega$ $\quad$ $p \not\equiv \pm 1 \pmod{\ell}$

Ex: $R^{ver}_{\bar{\rho}} = W(\bar{\mathbb{F}}_\ell)[[\alpha, \beta, c]]/(c(\alpha - \beta))$

$\rho^{ver}(\mathrm{Frob}) = \begin{pmatrix} 1+\alpha & 0 \\ 0 & p(1+\beta) \end{pmatrix}$.

$$\rho^{\mathrm{ver}}(\sigma) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$



So there really is no hope here. So we need to lower our expectations.

$$R_{\bar\rho}^{\mathrm{ver}} \subseteq W(\bar{\mathbb{F}}_\ell)[[\alpha,\beta,c]]/c \oplus W(\bar{\mathbb{F}}_\ell)[[\alpha,\beta,c]]/\alpha-\beta$$

$= (r_1, r_2) \qquad r_1 \equiv r_2 \quad (c, \alpha-\beta)$.

M over $W(\bar{\mathbb{F}}_\ell)[[\alpha,\beta,c]]/c$ principal series, generic quotient over $Z(\alpha-\beta)$.

N Steinberg w/ char. $\chi(\mathrm{Frob}) = 1+\alpha$ over $W(\bar{\mathbb{F}}_\ell)[[\alpha,\beta,c]]/(\alpha-\beta)$

$$f: M/(\alpha-\beta)M \twoheadrightarrow N/cN$$

pairs $(m,n) \in M \oplus N$, $f(m) = n$ in N/cN.

<u>Thm:</u> g wt 2 eigenform of level N, $p \nmid N$, then

$$\bar\rho_g|_{D_p} = 1 \oplus \omega$$

$$S = \lim_{r\to\infty} S_2(\Gamma(p^r) \cap \Gamma_1(N), W(\bar{\mathbb{F}}_\ell))$$

$m \subset \mathbb{T}$ corresponding to g, S_m is $\mathbb{T}_m[GL_2(\mathbb{Q}_p)]$-module

<u>Thm (Emerton):</u> A a reduced complete Noetherian local flat $W(\bar{\mathbb{F}}_\ell)$-algebra, $\rho: G_{\mathbb{Q}_p} \to GL_2(A)$. Then $\exists$ at most one $A[GL_2(\mathbb{Q}_p)]$-module π s.t.

1) π is "A-torsion free" (every associated prime of π is minimal)

2) at minimal primes x, π_x corresponds to ρ_x via L.L.C.

③ $\exists$ a surjection $\tilde{\pi}(\bar{\rho}) \to \pi/m\pi$.

<u>Conjecture (Emerton):</u> There always is such a module.

It seems feasible ~~that~~ the construction given here will give this conjecture.