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## Level lowering for p-adic modular forms:

$N \geq 1$ level

$X_1(N) := X(\Gamma_1(N))$.

$f \in H^0(X_1(N), \omega^{\otimes k})$ a weight $k$ newform with coefficients in $\overline{\mathbb{Q}}_p$.

$\lambda_f$ the system of eigenvalues (away from $\ell$)

Atkin-Lehner Theory: $f(q), f(q^\ell), f(q^{\ell^2}), \ldots$ span

$$\varinjlim_n H^0(X(\Gamma_1(N) \cap \Gamma_0(\ell^n)), \omega^{\otimes k})[\lambda_f].$$

Langlands-Deligne-Carayol:

$$\varinjlim_n H^0(X(\Gamma_1(N) \cap \Gamma(\ell^n)), \omega^{\otimes k})[\lambda_f]$$

$GL_2(\mathbb{Q}_\ell)$

$$= \pi(\rho_f|_{G_{\mathbb{Q}_\ell}})$$

via L.L.C.

$$\langle f(q), f(q^\ell), \ldots \rangle = \pi(\rho_f|_{G_{\mathbb{Q}_\ell}})^{\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}} \qquad (\ell \neq p).$$

Local Langlands: $(\ell \neq p)$

$\rho: G_{\mathbb{Q}_\ell} \to GL_2(K)$, $K = \mathrm{Frac}(A)$ where $A$ is a complete local domain, $p$-torsion free, res. char. $p$.

$\overset{\text{L.L.C.}}{\rightsquigarrow} \pi \circlearrowleft GL_2(\mathbb{Q}_\ell)$ smooth adm.

K. v.s.

Example: $\rho = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ Then $\pi = \mathrm{Ind}_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}^{GL_2(\mathbb{Q}_\ell)} \chi_1 |\cdot| \otimes \chi_2$.

If $\chi_1 \chi_2^{-1} = |\cdot|^{\pm}$, label s.t. $\chi_1 \chi_2^{-1} = |\cdot|$. (In other cases the ordering does not matter!)

$\rho = \chi_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & |\cdot|^{-1} \end{pmatrix}$. is an example of this, though this can not occur classically. Then we get

$$(\chi, \circ\det) \otimes \underbrace{\mathrm{Ind}\ \ |\cdot| \otimes |\cdot|^{-1}}_{0 \to St. \to \mathrm{Ind} \to 1\text{-dim} \to 0.}$$

This case is called non-generic.

If $\rho = \chi \otimes \begin{pmatrix} 1 & * \\ 0 & |\cdot|^{-1} \end{pmatrix}$ non-split, $\pi = (\chi\circ\det) \otimes St$

If $\rho$ irred, then $\pi$ is supercuspidal.

<u>Characteristic p L.L.C:</u>

<u>Thm:</u> There exist a unique association

$$\bar\rho : G_{\mathbb{Q}_\ell} \to GL_2(k) \quad \text{finite over } \mathbb{F}_p$$

$$\wr$$

$$\bar\pi(\bar\rho) \circlearrowleft GL_2(\mathbb{Q}_\ell)$$

s.t. • formation of $\bar\pi$ is compatible w/ extending scalars

- If $\rho : G_{\mathbb{Q}_\ell} \to GL_2(E)$ (finite$/\mathbb{Q}_p$) lifting $\bar\rho$, then letting

$\pi(\rho)^\circ$ denote the unique (up to non-zero scaling) lattice in $\pi(\rho)$ s.t. $\overline{\pi(\rho)^\circ}$ has generic socle.

$\exists$ embedding $\overline{\pi(\rho)^\circ} \hookrightarrow \bar\pi(\bar\rho)$.

• $\exists$ a lift $\rho$ (possibly after extending scalars) s.t.

$$\overline{\pi(\rho)^\circ} \xrightarrow{\sim} \bar\pi(\bar\rho).$$

<u>L.L in p-adic families:</u>

**Thm:** Let $A$ be a complete, reduced, Noeth. local ring, res. field $k$, finite $/\mathbb{F}_p$, $A$ $p$-torsion free. Given

$$\rho : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(A),$$

then there exist at most one (up to isom.) torsion-free $A$-module $V$ s.t.

1) If $\mathfrak{a} \in \operatorname{Spec} A$ is minimal, then
$$K(\mathfrak{a}) \otimes_A V \simeq \widetilde{\pi}(K(\mathfrak{a}) \otimes_A V)$$ ← smth contragredient.

2) $\exists$ non-zero surj.
$$\overline{\pi}(\rho / m\rho) \longrightarrow V/mV \qquad m = \text{max ideal of } A$$

**Conjecture:** Such a $V$ always exists.

**Remark:** This conjecture is probably a theorem of Helm.

If $V$ exists and $\wp \in \operatorname{Spec}(A[\frac{1}{p}])$ a closed pt., then $\exists$ a surjection

$$\widetilde{\pi(K(\wp) \otimes_A \rho)} \xrightarrow{\text{non-zero}} K(\wp) \otimes_A V.$$

↑ usually, but not always, an isom.

**p-adic modular forms:**

$E/\mathbb{Q}_p$ finite $\quad \mathcal{O} \subseteq E$,

$$\begin{tikzcd} Ig_r(\Gamma_1(N) \cap \Gamma(\ell^n)) / \mathcal{O}/\varpi^s\mathcal{O} \arrow[d, "(\mathbb{Z}/p^r\mathbb{Z})^\times"'] \\ X(\Gamma_1(N) \cap \Gamma(\ell^n))^{\text{ord}} / \mathcal{O}/\varpi^s\mathcal{O} \end{tikzcd}$$

Igusa tower

$$\hat{\mathcal{M}}(\Gamma_1(N)\cap\Gamma(\ell^r), \mathcal{O}) = \varprojlim_s \varinjlim_r H^0(\mathcal{I}g_r/\mathcal{O}/\varpi^s\mathcal{O}, \mathcal{O}_{\mathcal{I}g_r})$$

$$\hat{\mathcal{M}}(\Gamma_1(N)\cap\Gamma(\ell^\infty), \mathcal{O}) = \varinjlim_r \hat{\mathcal{M}}(\Gamma_1(N)\cap\Gamma_1(\ell^r), \mathcal{O})$$

$GL_2(\mathbb{Q}_\ell)$ $\circlearrowleft$ $\mathbb{T} = \mathcal{O}[\![\mathbb{Z}_p^\times]\!][T_q, q\langle q\rangle, U_p]$ large Hecke alg.

Let $m$ be a maximal ideal in $\mathbb{T}$ corresponding to $\bar{\rho}: G_\mathbb{Q} \to GL_2(\mathbb{F}_q)$ abs. irred, can form

$$\hat{\mathcal{M}}(\Gamma_1(N)\cap\Gamma(\ell^\infty), \mathcal{O})_m$$

$$\curvearrowleft\ \mathbb{T}_m[GL_2(\mathbb{Q}_\ell)]$$

$$\rho^{mod}: G_\mathbb{Q} \to GL_2(\mathbb{T}_m).$$

<u>Thm:</u> The $V$ for $\rho^{mod}|_{G_{\mathbb{Q}_\ell}}$ exists, and

$$\hat{\mathcal{M}}_m \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}\text{-lin cont}}(V, \mathcal{O})_{smooth}.$$

<u>Cor:</u> If $\wp \in \mathrm{Spec}\,\mathbb{T}_m[\frac{1}{p}]$ is closed, then

$$\hat{\mathcal{M}}_m[\wp] \longrightarrow \pi(K(\wp)\otimes_{\mathbb{T}_m} \rho|_{G_{\mathbb{Q}_\ell}})^\circ$$

<u>Thm:</u> Suppose either

1) $p>2$, $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is abs. irred (w/ Kisin)

2) $\bar{\rho}|_{G_{\mathbb{Q}_p}} \not\sim$ twist of $\begin{pmatrix}1 & * \\ 0 & \text{cyclo}\end{pmatrix}$

then the injection in the previous cor. is an isom.