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Non-trivial extensions of p-adic Galois representations that are trivial at p:

I Extensions:

K = number field

$\rho: G_K \to GL_n(L)$. There are two cases we are interested in, $L/\mathbb{F}_p$ or $L/\mathbb{Q}_p$. We are mostly interested in $L/\mathbb{Q}_p$.

An extension of 1 by ρ is a U s.t.

$$0 \to \rho \to U \to 1 \to 0.$$

Then $[U] \in H^1(K, \rho)$. Assume ρ is geometric.

Local properties: Let v be a finite place of K, $D_v \subset G_K$ the decomposition group.

No condition	nothing	ϕ
trivial	$U\vert_{D_v} = 1 \oplus \rho\vert_{D_v}$	0
good reduction	if $v \nmid p$, $0 \to \rho^{I_v} \to U^{I_v} \to 1 \to 0$ is exact. if $v \mid p$, then $0 \to D_{crys}(\rho\vert_{D_v}) \to D_{crys}(U\vert_{D_v}) \to 1 \to 0$ is exact.	f.

$H^1_{\mathcal{Z}}(K, \rho) = \{[U] \in H^1(K,\rho)$ s.t. if $v \mid p$, then U is $\mathcal{Z}(v)$ at v and if $v \nmid p$, v is $f\}$.

$\mathcal{Z}: \{\text{places dividing } p\} \longrightarrow \{0, \phi, f\}$

Fix a place v dividing p.

$$0 \Rightarrow f \Rightarrow \phi$$

if Hodge-Tate wt of $\rho|_{D_v}$ are >0 then $\Leftarrow$

if HT wt <-1 then $\Leftarrow$

(1)

ρ should have motivic weight w. We do not consider $w=-1$ in this talk.

	$w(\rho) < -1$	$w(\rho) > -1$	
$\dim H^1_0(K,\rho)$	0	0	(Jannsen conj.)
$\dim H^1_f(K,\rho)$	$\sum_{v\mid p} \dim H^1_f(D_v,\rho) - \sum_{v\mid\infty} \dim \rho^{D_v} + \dim \rho^{G_K}$	0	(Bloch-Kato conj.)
$\dim H^1_\phi(K,\rho)$	$nd - \sum_{v\mid\infty} \dim \rho^{D_v} + \dim \rho^{G_K}$	same here as	(Jannsen conj.)

$nd = (\dim \rho)[K:\mathbb{Q}]$

$\mathbb{Q}_p(n)$: Bloch-Kato is known (Soulé)

For line 3 it is not known.

For line 1, we have $\geq$.

If $\rho = 1$, $\dim H^1_\phi(K,\rho) = d - (r_1+r_2) + 1$

$= r_2 + 1$ (Leopoldt conjecture)

Question: What is $\dim H^1_?(K,\rho) = ?$

Related question:

$$H^1(K,\rho) \hookrightarrow \prod_{v\mid p} H^1(D_v,\rho)$$

$$H^1_f(K,\rho) \hookrightarrow \prod_{v\mid p} H^1_f(D_v,\rho)$$

In the case $H^1_f(K,\mathbb{Q}_p(1)) \longrightarrow \prod_{v\mid p} H^1_f(K,\mathbb{Q}_p(1))$.

$$\begin{array}{ccc} H^1_f(K,\mathbb{Q}_p(1)) & & \\ \wr\Vert & & \\ \mathcal{O}_K^\times \otimes \mathbb{Q}_p & \longrightarrow & \prod (\mathcal{O}_{K_v}^\times \otimes \mathbb{Q}_p) \end{array}$$