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Eisenstein ideals and main conjectures in Iwasawa theory:

p odd prime, fix embeddings $\overline{\mathbb{Q}} \overset{i_\infty}{\hookrightarrow} \mathbb{C}$, $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p$

ω = Teichmüller character

$\omega : \mathbb{Z}_p^\times \longrightarrow \mu_{p-1} \subset \mathbb{Z}_p^\times$.

$\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}$ Dir. char., $p^n = \mathrm{cond}(\chi)$.

$k \geq 2$, $\chi(-1) = (-1)^k$, $M_k(\chi, \mathbb{C})$ = modular forms of weight k and character χ, i.e.,

$$f : \mathfrak{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \} \to \mathbb{C}$$

holomorphic and $(f|_k\gamma)(z) = \chi(d) f(z)$ $\forall\, \gamma \in \Gamma_0(p^n)$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, define $(f|_k\gamma)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$.

The fact that $f(z+1) = f(z)$ $\forall\, z \in \mathfrak{H}$ $\Rightarrow$ f has a Fourier exp.

$f(z) = \sum_{m \in \mathbb{Z}} a_m(f) q^m$, $q = e^{2\pi i z}$, $a_m = 0$ for $m < 0$. $\leftarrow$ holomorphic at ∞.

f is a cusp form if $a_0(f|\gamma) = 0$ $\forall\, \gamma \in SL_2(\mathbb{Z})$.

The space of cusp forms is denoted $S_k(\chi, \mathbb{C})$.

The sets $S_k(\chi, \mathbb{C})$ and $M_k(\chi, \mathbb{C})$ are f.d. $\mathbb{C}$-v.s.'s.

For $A \subset \mathbb{C}$, containing values of χ, write $S_k(\chi, A)$, $M_k(\chi, A)$ are the forms w/ $a_n(f) \in A$ $\forall n$.

Hecke operators:

These are operators that act on these spaces

$\forall\, \ell \neq p$: T_ℓ : $(f|T_\ell)(z) = \sum_{a=0}^{\ell-1} f\left(\frac{z+a}{\ell}\right) + \chi(\ell) f(\ell z)$

$\ell = p$ U_p : $(f|U_p)(z) = \sum_{a=0}^{p-1} f\left(\frac{z+a}{p}\right)$.

If f is an eigenform for all Hecke operators, and if $a_1(f)=1$

$$f|T_\ell = a_\ell(f) f$$

$$f|U_p = a_p(f) f$$

then we say f is normalized.

$$S_k(\chi, \bar{\mathbb{Q}}) \otimes \mathbb{C} \simeq S_k(\chi, \mathbb{C}).$$

<u>Example:</u> Eisenstein series: $\Gamma_\infty = \{ \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \}$, $k>2$ or $k=2$ and χ nontrivial

$$\mathcal{E}_{k,\chi}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(p^n)} (cz+d)^{-k} \chi(d)$$

After a suitable normalization,

$$E_{k,\chi}(z) = \frac{L(1-k,\chi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n$$

where $\sigma_{k-1,\chi}(n) = \sum_{\substack{d|n \\ (d,p)=1}} d^{k-1}\chi(d).$

$$L(s,\chi) = \sum_{m=1}^{\infty} \chi(m) m^{-s}.$$

$L(1-k,\chi) \in \bar{\mathbb{Q}}$ if $k \geq 2$, $(-1)^k = \chi(-1)$.

If $\chi = 1$ and $k \not\equiv 0 \pmod{p-1}$, then $L(1-k,\chi)$ is p-adic integer.

<u>Hida Theory:</u>

$\Lambda = \mathbb{Z}_p[[\Gamma]]$ $\quad \Gamma = 1+p\mathbb{Z}_p \ni u$ = top. gen, $\quad \Delta = \mu_{p-1} \subseteq \mathbb{Z}_p^\times$.

$\Lambda \simeq \mathbb{Z}_p[[T]]$ where $1+T \longleftrightarrow u$,

ψ finite order character of Γ, $k \geq 2$,

$\psi_k : \Lambda \longrightarrow \bar{\mathbb{Q}}_p$ and then extend this.

$u \longmapsto \psi(u)\, u^k$

<u>Λ-adic form:</u>

$\mathbb{F} \in \Lambda[[q]]$

$\mathbb{F}(q) = \sum_{m=0}^{\infty} a_m(\mathbb{F}) q^m$

$\psi_k(\mathbb{F}) = \sum_{m=0}^{\infty} \psi_k(a_m) q^m \in \bar{\mathbb{Q}}_p[[q]]$

$\mathbb{F}$ is a Λ-adic form iff $\forall\, k \geq 2,\ \forall \psi,\ \psi_k(\mathbb{F}) = i_p \circ i_\infty^{-1}(\mathbb{F}_{k,\psi})$

with $\mathbb{F}_{k,\psi} \in M_k(\psi\omega^{-k}\chi_0, \mathbb{C})$ where $\chi_0 : \Delta \to \mu_{p-1}$

χ_0 = nebentypus of $\mathbb{F}$, i.e., $\mathbb{F}$ specializes to modular forms at k, ψ.

T_ℓ and U_p act on Λ-adic forms.

$F \in M_k(\chi, \bar{\mathbb{Q}}_p) \circlearrowleft U_p$

$$e = \lim_{n\to\infty} U_p^{n!}, \quad e^2 = e \qquad \text{(Hida's ordinary projector)}$$

We say F is ordinary if $F|e = F$.

$M_k^{ord} = e M_k$

$S_k^{ord} = e S_k$.

$\mathbb{F}$ a Λ-adic form, we say $\mathbb{F}$ is ordinary if $\mathbb{F}_{k,\psi}$ is ordinary

$\forall k, \psi$.

We denote the space of ordinary Λ-adic forms of nebentypus χ_0 by $M^{ord}_{\chi_0}(\Lambda)$, similarly for $S^{ord}_{\chi_0}(\Lambda)$,

$$S^{ord}_{\chi_0}(\Lambda) = \{ \mathbb{F} \in M^{ord}_{\chi_0}(\Lambda) : a_0(\mathbb{F}) = 0 \}.$$

Note we only need to worry about other cusps if there is some tame level, which we are not working with here.

<u>Thm (Hida 80's):</u>

① $$M^{ord}_{\chi_0} \otimes_{\psi_k} \overline{\mathbb{Q}}_p \xrightarrow{\sim} M^{ord}_k(\psi\omega^{-k}\chi_0, \overline{\mathbb{Q}}_p) \qquad \forall k \geq 2, \psi$$

$$S^{ord}_{\chi_0} \otimes_{\psi_k} \overline{\mathbb{Q}}_p \xrightarrow{\sim} S^{ord}_k(\psi\omega^{-k}\chi_0, \overline{\mathbb{Q}}_p)$$

Moreover, $M^{ord}_{\chi_0}$ and $S^{ord}_{\chi_0}$ are free of finite rank over Λ.

② $$0 \to S^{ord}_{\chi_0} \to M^{ord}_{\chi_0} \to \Lambda \to 0. \quad \text{is exact.}$$
$$\mathbb{F} \longmapsto a_0(\mathbb{F})$$

(again, no tame level here!) (χ_0 = even character)

Hida proved this theorem trying to make an Iwasawa theory of modular forms.

<u>Example:</u>

Kubota-Leopoldt p-adic L-function $\longleftarrow$ p-adic L-function

$\chi_0 : \Delta \to \mu_{p-1}$, $\quad F_{\chi_0} \in \mathrm{Frac}(\Lambda)$

If $\chi_0 \neq \omega^{-1}$, $\mathcal{F}_{\chi_0} \in \Lambda \simeq \mathbb{Z}_p[\![T]\!]$

if $\chi_0 = \omega$, $T \cdot \mathcal{F}_{\chi_0} \in \Lambda^\times \simeq \mathbb{Z}_p[\![T]\!]^\times$

$\Psi_k(\mathcal{F}_{\chi_0}) = L^{(p)}(1-k, \chi_0 \psi \omega^{1-k})$

If $\gcd(d,p) = 1$, $\langle d \rangle_\Gamma = \omega^{-1}(d) d \in 1 + p\mathbb{Z}_p = \Gamma \subset \Lambda$.

In the case $\Lambda = \mathbb{Z}_p[\![T]\!]$, $\langle d \rangle_T = (1+T)^{\frac{\log_p(d\omega^{-1}(d))}{\log_p u}}$.

$$\sigma_{T,\chi_0}(n) = \sum_{\substack{d \mid n \\ \gcd(d,p)=1}} \langle d \rangle_T d^{-1} \chi_0(d).$$

Assume $\chi_0 \neq \omega^{-1}$.

$$\mathbb{E}_{\chi_0} := \frac{1}{2} \mathcal{F}_{\chi_0} + \sum_{n=1}^{\infty} \sigma_{T,\chi_0}(n) q^n \in \Lambda[\![q]\!]$$

$$\Psi_k(\mathbb{E}_{\chi_0}) = E_{k, \chi_0 \psi \omega^{1-k}}$$

$\Rightarrow$ $\mathbb{E}_{\chi_0}$ is a Λ-adic form of nebentypus χ_0

<u>Note:</u> The $E_{k,\chi}$ are all ordinary eigenforms.

$$E_{k,\chi} | U_p = E_{k,\chi}$$

$$E_{k,\chi} | T_\ell = (1 + \chi(\ell)\ell^{k-1}) E_{k,\chi}.$$

$\Rightarrow$ $\mathbb{E}_{\chi_0}$ is an eigenform and has U_p ev. 1,

T_ℓ e.v $1 + \langle \ell \rangle_T \ell^{-1} \chi_0(\ell)$.

$h^{ord}_{\chi_0}$ Λ-subalgebra of $\mathrm{End}_\Lambda(S^{ord}_{\chi_0})$ generated by the image of the T_ℓ's and U_p.

One can show

$$\begin{tikzcd} h^{ord}_{\chi_0} \otimes_{\psi_k} \overline{\mathbb{Z}}_p = h^{ord}_k(\psi\omega^{-k}\chi_0) \arrow[d, "\lambda"] \arrow[r, hook] & S^{ord}_k(\chi_0\omega^{-k}\psi, \overline{\mathbb{Q}}_p) \\ \overline{\mathbb{Z}}_p & \end{tikzcd}$$

Form $\sum_{m=1}^{\infty} \lambda(T_m) q^m$ = q-expansion of an eigenform of

More generally, given a character

$$\begin{tikzcd} h^{ord}_{\chi_0} \arrow[r, "\lambda"] & \overline{\mathbb{Z}}_p \\ \Lambda \arrow[u] \arrow[ur, "\psi_k"'] & \end{tikzcd}$$

we say λ is arithmetic when if we restrict to Λ we get ψ_k for some ψ and k.

$$\begin{tikzcd} \mathrm{Spec}(h^{ord}_{\chi_0})(\overline{\mathbb{Z}}_p) = \mathrm{Hom}(h^{ord}_{\chi_0}, \overline{\mathbb{Z}}_p) \arrow[d] & \ni x \arrow[d, mapsto] \\ \mathrm{Spec}(\Lambda) & \psi_k \end{tikzcd}$$

$h^{ord}_{\chi_0}$ is semisimple

$H^{ord}_{\chi_0\omega} \subset \mathrm{End}(M^{ord}_{\chi_0\omega})$ Hecke alg.

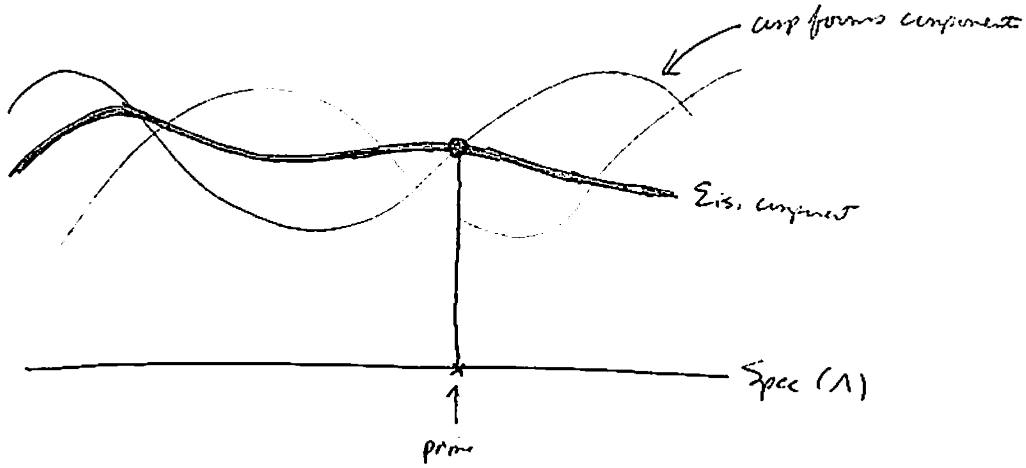
$H^{ord}_{\chi_0\omega} \xrightarrow{\lambda_{Eis}} \Lambda \longleftarrow$ irred component of $\mathrm{Spec}(H)$

Eisenstein component

$T_\ell \longmapsto 1 + \sigma_T(\ell)\,\ell^{-1}\omega\chi_0(\ell)$

$U_p \longmapsto 1$

Geometrically



This says at the intersection, we have a congruence between the Eisenstein series and a cusp form at the prime in $\mathrm{Spec}(\Lambda)$ it is over. It corresponds to a zero of $\mathcal{F}_{\chi_0}$.

$$\begin{array}{ccc} H^{ord}_{\chi_0\omega} & \hookrightarrow & h^{ord}_{\chi_0\omega} \times \Lambda \\ \cap & & \cap \\ \prod \mathbb{I}_i \times \Lambda & & \prod \mathbb{I}_i \times \Lambda \end{array}$$

Eisenstein component:

$$\mathrm{Spec}(\Lambda) \xhookrightarrow{\lambda^*_{Eis}} \mathrm{Spec}(H^{ord}_{\chi_0\omega})$$

$$\cup$$

$$\mathrm{Spec}(h^{ord}_{\chi_0\omega})$$

$$\mathrm{Spec}(h^{ord}_{\chi_0\omega} \otimes_{H^{ord}_{\chi_0\omega}} \Lambda)$$

|| Eisenstein quotient

$$h^{ord}_{\chi_0\omega}/I_{\chi_0\omega}$$

where $I_{\chi_0\omega}$ = ideal of $h^{ord}_{\chi_0\omega}$ generated by $T_\ell - \lambda_{Eis}(T_\ell)$ $\ell \neq p$ = Eisenstein ideal
$U_p - 1$

Thm: ($\chi_0 \neq \omega^{-1}$) $h^{ord}_{\chi_0\omega}/I_{\chi_0\omega} \xrightarrow{\sim} \Lambda/\mathcal{F}_{\chi_0}$.

Proof: Recall

$$0 \to S^{ord}_{\chi_0\omega} \to M^{ord}_{\chi_0\omega} \to \Lambda \to 0$$

$$\mathbb{F} \longmapsto a_0(\mathbb{F})$$

is exact.

So $\exists\ g \in M^{ord}_{\chi_0\omega}$: $a_0(g) = 1$

$$\mathbb{H} = \mathbb{E}_{\chi_0\omega} - \mathcal{F}_{\chi_0}\, g$$

$$\Rightarrow a_1(\mathbb{H}) \equiv 1 \pmod{m_\Lambda} \in \Lambda^\times.$$

$$\mathfrak{h}^{ord}_{\chi_0 \omega} \longrightarrow \Lambda/\mathfrak{P}_{\chi_0}$$

$$\begin{array}{c}\rotatebox{90}{$\in$} \\ T\end{array} \longrightarrow \frac{a_1(1, \mathbb{H}|T)}{a_1(1, \mathbb{H})} = \lambda_{Eis}(T)$$

$\Rightarrow \mathbb{H} \equiv E_{\chi_0} \pmod{\mathfrak{P}_{\chi_0}}$

This gives that the map is surj. $\blacksquare$

<u>Galois representations</u>:

$G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

$\forall \ell$, geometric Frobenius $Frob_\ell \in G_{\mathbb{Q}}$.

I_ℓ inertia group.

<u>Thm (Eichler-Shimura, Deligne)</u>: f Hecke eigenform of weight k and nebentypus χ. Then $\exists$ cont. homom. $\rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ s.t.

- ρ_f unram away from p
- $\det(1 - \rho_f(Frob_\ell)X) = 1 - a_\ell(f)X + \chi(\ell)\ell^{k-1}X^2$ $\forall \ell \neq p$.

If moreover, f is ordinary, then

$$\rho_f|_{D_p} \simeq \begin{pmatrix} \delta_f & * \\ 0 & \delta_f^{-1}\chi\varepsilon^{1-k} \end{pmatrix}$$

ε = cyclotomic char. and δ_f = unram. char s.t.
$\delta_f(Frob_p) = \alpha_p$ where α_p is unit root of char. poly. or
$\alpha_p = a_p$ if $\chi \neq \mathbb{1}$ ...

Example: $f = E_{k,\chi}$.

$$\rho_f = \begin{pmatrix} 1 & 0 \\ 0 & \chi\varepsilon^{1-k} \end{pmatrix}$$

Thm: If f is cuspidal, then ρ_f is absolutely irred.

Sketch: $L(f,\chi,s) =$ entire function

if ρ_f were reducible

$$\rho_f \simeq \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \qquad \chi_1, \chi_2 \text{ alg. chars}$$

$$\Rightarrow L(f \otimes \chi_1^{-1}, s) = \zeta(s)\, L(\chi_2\chi_1^{-1}, s).$$

Then $s=1$ gives a pole, #.

Using the theory of pseudo-representations one can show that given a Hida family; one has an assoc. Galois rep:

$$H^{ord}_{\chi_0\omega} \xrightarrow{\lambda_{\mathbb{I}}} \mathbb{I} \quad \text{some irred component of } H^{ord}_{\chi_0} \qquad F_{\mathbb{I}} \text{ fraction field of } \mathbb{I}$$

$$\rightsquigarrow \rho_{\mathbb{I}} : G_{\mathbb{Q}} \longrightarrow GL_2(F_{\mathbb{I}}).$$

$$\det(1 - \rho_{\mathbb{I}}(\mathrm{Frob}_\ell)\, x) = 1 - \lambda_{\mathbb{I}}(T_\ell)x + \chi_0\omega(\ell)\langle \ell \rangle_T^{-1} \ell \qquad \ell \neq p$$

$$\rho_{\mathbb{I}}|_{I_p} \simeq \begin{pmatrix} 1 & * \\ 0 & \chi_0\omega\varepsilon^{-1}\langle \ell \rangle_T \end{pmatrix}$$

$$\rho_{\mathbb{E}_{\chi_0}} = \begin{pmatrix} 1 & 0 \\ 0 & \langle \varepsilon \rangle_{\mathcal{F}} \kappa^{-1} \chi_0 \end{pmatrix} \qquad \kappa = \varepsilon \omega^{-1}.$$

R_{Eis} = local component of $h^{ord}_{\chi_0 \omega}$ corresponding to the maximal ideal of $h^{ord}_{\chi_0\omega}$ containing $I_{\chi_0 \omega}$

$$R_{Eis}/I_{\chi_0\omega} \longrightarrow \Lambda/\mathcal{F}_{\chi_0}$$

For each irreducible component of $R_{Eis} \rightsquigarrow \rho_{\mathrm{II}}$

$$\rho_{R_{Eis}} : G_{\mathbb{Q}} \longrightarrow GL_2(\tilde{R}_{Eis})$$

$\tilde{R}_{Eis}$ = Fraction ring of R_{Eis}.

$$\underset{\substack{\uparrow \\ \text{irred rep.}}}{tr(\rho_{R_{Eis}})} \equiv \underset{\substack{\uparrow \\ \text{red. rep.}}}{tr(\rho_{\mathbb{E}_{\chi_0}})} \mod I_{\chi_0\omega}$$

$\mathcal{L} \subset V_{\tilde{R}_{Eis}} = \tilde{R}_{Eis}^{2}$ ← w/ action given by $\rho_{R_{Eis}}$

We can use this congruence to construct a lattice $^\vee$ with nice properties s.t.

$$0 \to \mathcal{N}(\chi_0 \langle \varepsilon \rangle \kappa^{-1}) \longrightarrow \mathcal{L}/I_{\chi_0\omega}\mathcal{L} \longrightarrow R_{Eis}/I_{\chi_0\omega} \longrightarrow 0$$

with ↓ having no quotient with action given by $\chi_0 \langle \varepsilon \rangle \kappa^{-1}$.

where $\mathcal{N}$ is some torsion Λ-module s.t. $\mathrm{char}_\Lambda(\mathcal{N})$ is divisible by $\mathcal{F}_{\chi_0}$.

Choose in $V_{\tilde{R}_{Eis}} = (\tilde{R}_{Eis})^2 = \left(\prod_{\mathbb{I}} F_{\mathbb{I}}\right)^2$ an element v^+

s.t. $\rho_{R_{Eis}}(c)\, v^+ = v^+$ with nontrivial projection on any component.

$\mathcal{L} =$ $R_{Eis}[G_{\mathbb{Q}}]$-module generated by v^+.

$\Rightarrow \mathcal{L}$ is a lattice $\mathcal{L} \otimes F_\Lambda \simeq V_{\tilde{R}_{Eis}}$ (use irreducibility here)

$\mathcal{L} = \mathcal{L}^- \oplus \mathcal{L}^+$ s.t. $\mathcal{L}^+$ = fixed part of $\mathcal{L}$ by $\rho_{\hat{R}_{Eis}}(c)$

$\mathcal{L}^- =$ (-1)-eigenspace (c = complex conj)

$\sigma \in R_{Eis}[G_{\mathbb{Q}}]$.

$$\rho_{\mathcal{L}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \qquad \begin{array}{l} a_\sigma \in \mathrm{Hom}(\mathcal{L}^-, \mathcal{L}^-) \\ b_\sigma \in \mathrm{Hom}(\mathcal{L}^+, \mathcal{L}^-) \\ \vdots \end{array}$$

$$a_\sigma + d_\sigma = \mathrm{tr}(\rho_{\mathcal{L}}(\sigma)) \in R_{Eis} \qquad \forall \sigma \in R_{Eis}[G_{\mathbb{Q}}]$$

$$\text{III} \quad 1 + \chi_0 \kappa^{-1} \langle \varepsilon_T \rangle(\sigma) \pmod{I_{\chi_0 \omega}}$$

$$\Rightarrow a_\sigma \equiv \chi_0 \kappa^{-1} \langle \varepsilon \rangle(\sigma) \pmod{I_{\chi_0 \omega}} \quad \text{(look at c.c)}$$

$$d_\sigma \equiv 1$$

$$a_{\sigma\tau} = a_\sigma a_\tau + b_\sigma c_\tau$$

$\Rightarrow b_\sigma c_\tau \in I_{\chi_0 \omega}$ $\forall \sigma, \tau$ and $c_\tau b_\sigma \in I_{\chi_0 \omega}$ $\rightsquigarrow c_\tau(\mathcal{L}^-) \subset I_{\chi_0 \omega} v^+ = I_{\chi_0 \omega} \mathcal{L}^+$

$\Rightarrow a_\sigma, d_\sigma \in R_{Eis}$

Any element of $\mathcal{L}$ is of the form $\quad v^+ = \binom{0}{v^+}$

$$\rho_{\mathcal{L}}(\sigma) \cdot v^+ = \underset{\mathcal{L}^-}{\underset{\wedge}{b_\sigma v^+}} + \underset{\mathcal{L}^+}{\underset{\wedge}{d_\sigma v^+}} \quad \Rightarrow$$

$\Rightarrow \mathcal{L}^+ = R_{Eis} v^+$ and so is free of rank 1 over R_{Eis}.

The action of σ on $\mathcal{L}/I\mathcal{L}$ fits into an exact sequence

split after restriction to I_p

$$\chi_0 \kappa^{-1} \langle \varepsilon \rangle_+$$

$$0 \to \mathcal{L}^- / I_{\chi_0 \omega} \mathcal{L}^{-1} \; (\ast) \to \mathcal{L}/I\mathcal{L} \to \mathcal{L}^+/I\mathcal{L}^+ \to 0$$

$$\mathcal{L}^+/I\mathcal{L}^+ \simeq \Lambda / \mathcal{T}_{\chi_0}$$

Characteristic ideal is divisible by $\mathcal{T}_{\chi_0}$

$$\mathcal{N}^* = \mathrm{Hom}_\Lambda(\mathcal{N}, \Lambda^*) \quad \text{with } \Lambda^* = \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$$

$$= \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{N}, \mathbb{Q}_p/\mathbb{Z}_p)$$

$$\mathcal{N}^* \hookrightarrow H^1(G_\mathbb{Q}, \Lambda^*(\chi_0 \kappa^{-1} \langle \varepsilon \rangle_+))$$

$$\varphi \longmapsto \varphi \circ c_\mathcal{N}$$

<u>Remark:</u> M $\Lambda[G_\mathbb{Q}]$-module

$H^1(G_\mathbb{Q}, M)$ classifies the extensions of the form

$$0 \to M \to E \to \Lambda \to 0$$

$$c_\mathcal{N} : G_\mathbb{Q} \to \mathcal{N}$$

$\forall\, \ell \neq p$, $\rho_\mathcal{L}$ is unram at ℓ

For $\ell = p$, $\rho_\mathcal{L}|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & \langle \varepsilon \rangle \chi_0 \kappa \end{pmatrix}$ $\qquad \chi_0 \neq 1 \mod \mathfrak{m}_{R_{Eis}}$

$$\Rightarrow \quad \varphi \circ c_N \big|_{I_p} = 0 .$$

$$Sel(\chi_0) = \ker \left(H^1(G_{\mathbb{Q}}, \Lambda^*(\chi_0 \langle \varepsilon \rangle_\gamma \kappa)) \to \bigoplus_{\ell} H^1(I_\ell, -) \right)$$

$$\Rightarrow \quad Sel(\chi_0)^* \longrightarrow N$$

$\Rightarrow$ char ($Sel(\chi_0)^*$) is divisible by $\mathcal{F}_{\chi_0}$

related to $X_\infty = Gal(L_\infty / K_\infty)$

$$\forall \chi_0 \neq \omega^{-1}, \qquad \mathcal{F}_{\chi_0} \mid \mathcal{G}_{\chi_0} = \text{char ideal related to } X_\infty$$

on the other hand

$$\prod_{\substack{\chi_0 \text{ odd} \\ \chi_0 \neq \omega^{-1}}} \mathcal{F}_{\chi_0} \sim \prod_{\substack{\chi_0 \neq \omega^{-1} \\ \chi_0 \text{ odd}}} \mathcal{G}_{\chi_0} .$$

$$\Rightarrow \qquad \mathcal{F}_{\chi_0} \sim \mathcal{G}_{\chi_0} .$$

We now would like to generalize the proof of the main conjecture.

Let F be a number field, G_F the absolute Galois group. $\mathcal{O}/\mathbb{Z}_p$,

$T = T_p$ free $\mathcal{O}$-module of finite type, $\rho : G_F \to GL(T_p)$.

Def: If $E/\mathbb{Q}_p$, $V = L$-v.s., $L = \mathrm{Frac}(\mathcal{O})$, $\rho : G_E \to GL(V)$

we say ρ is ordinary iff

$\exists$ filtration F^iV of V such that $F^{i+1}V \subset F^i V$, $F^n V = 0$ for $n \gg 0$, $F^{-n}V = V$, such that $F^iV/F^{i+1}V \circlearrowleft I_E$ by ε^i

$$\underset{\displaystyle Gr^iV}{\overset{\parallel}{F^iV/F^{i+1}V}}$$

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where ε = cyclotomic char.

· If $Gr^iV \neq 0$ we say that $-i$ is a Hodge-Tate weight of V.

We say $\rho : G_E \to GL(T_p)$ is ordinary if $\forall v | p$ place of F, if $\rho|_{D_v}$ is ordinary.

$\forall v | p$, $F^i_v T_p$ s.t. $F^i_v T_p / F^{i+1}_v T_p \circlearrowleft^{I_v}$ by ε^i.

<u>Example:</u> f modular form of weight $k \geq 2$ of level prime to p, s.t. $p \nmid a(p,f)$ then

$$\rho_f\Big|_{I_p} \simeq \begin{pmatrix} 1 & * \\ 0 & \varepsilon^{1-k} \end{pmatrix}.$$

If V_f is the rep. space, $F^i = V_f$ if $i \leq 1-k$,

F^i = unramified line if $0 \geq i \geq 1-k$

$F^i = 0$ if $i > 0$.

Thus, $0, k-1$ are the Hodge-Tate weights.

ordinary

$V_p = T_p \otimes L$, $H^1(F, V_p)$ classifies the extensions of the form

$$0 \to V_p \to E \to L \to 0.$$

(potentially ordinary means $\rho|_{G_{F'}}$ is ordinary for some finite ext F'/F)

$Sel(F,\rho) \subset H^1(F, V_\rho)$. The Selmer group classifies the extensions which are ordinary and condition at places outside p. This is for characteristic zero. For torsion we need something different.

Let Σ = finite set of places of F. ρ ordinary,

$$Sel^{\Sigma}(F, V_\rho/T_\rho) \subset H^1(F, V_\rho/T_\rho)$$

$$\| $$

$$\lim_{n\to\infty} \left(Sel^{\Sigma}(F, p^{-n}T_\rho/T_\rho) \right) \leftarrow \text{classifies extensions}$$

$$0 \to p^{-n} T_\rho/T_\rho \to E \to p^{-n}\mathcal{O}/\mathcal{O} \to 0$$

such that for all $v \nmid p$, $v \notin \Sigma$

$$0 \to (p^{-n}T_\rho/T_\rho)^{I_v} \to E^{I_v} \to p^{-n}\mathcal{O}/\mathcal{O} \to 0 .$$

if $v | p$, $E|_{I_v} \in H^1_{ord}(F_v, p^{-n}T_\rho/T_\rho)$ classifies extensions which are obtained as restriction mod p^n of ordinary representations.

Given

$$e \longmapsto p^{-n}$$

$$0 \to p^{-n} T_\rho/T_\rho \to E \to p^{-n}\mathcal{O}/\mathcal{O} \to 0$$

$F_v^i E \rightsquigarrow I_v$ acts on $F_v^i E/F_v^{i+1}E$ by ε^i

$\mathcal{O}/p^n\mathcal{O} + F^n E$ is stable by I_v

$\|$

$\mathcal{O}/p^n\mathcal{O}\, e + F^n p^{-n}T_\rho/T_\rho$

$\Rightarrow$ the image of the class of E in $H^1\left(I_v, p^{-n}T_\rho/T_\rho \big/ p^{-n}F_v^n T_\rho/F_v^n T_\rho\right)$ is trivial.

Example: $\rho = \varepsilon^{-n}$ $n \geq 0 \Rightarrow F^{\circ} = 0 \Rightarrow$ the condition for the cocycle $H^1(F, \mathbb{Q}_p/\mathbb{Z}_p(\varepsilon^{-n}))$ is unramified $\forall v | p$. If $n < 0$, there is no condition.

K_∞/K $\mathbb{Z}_p^d$ - ext.

$$Sel^{\Sigma}(K_\infty, V_\rho/T_\rho) := \varinjlim_n Sel^{\Sigma}(K_n, V_\rho/T_\rho).$$

Using Shapiro's lemma

$$\subset H^1(F, T_\rho \otimes \Lambda^*)$$

where $\Lambda = \mathbb{Z}_p[[T]]$, $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$.

One defines $Sol^{\Sigma}(F, T_\rho \otimes \Lambda^*)$ in the same way.

This is why Hida theory is useful, you get something in

rather than constructing something at each level.

Any $\mathbb{I} \cdot \mathbb{Z}_p[[T_1, \ldots, T_r]]$. $\forall$ $V_{\mathbb{I}}$ rep. of G_F

ordinary, $\mathrm{Spec}(\mathbb{I})$ + can define Selmer group.

Construction of elements in "general" Selmer group:

The main idea is to deform reducible representations.

$\mathcal{O}$ = local Noetherian reduced ring

$\rho_0 = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r$ $\rho_i : G_F \to GL_{n_i}(\mathcal{O})$. $\overline{\rho_i} \neq \overline{\rho_j}$ $i \neq j$.

Assume that $\rho_i \bmod m_{\mathcal{O}}$ is abs. irred.

A big deformation of ρ_0 is given by (ρ, R, I) where (1) R is a local Noetherian reduced $\mathcal{O}$-algebra, (2) $I \subsetneq R$ ideal,

(3) $\rho : G_F \to GL_n(\mathrm{Frac}(R))$, $n = n_1 + \cdots + n_s$.

(4) $tr(\rho) \subset R$, $tr(\rho) \equiv tr(\rho_0) \bmod I$.

(5) $tr\,\rho \neq tr(\rho_1') + \cdots + tr(\rho_s')$ where the ρ_i''s are deformations of ρ_i.

(i.e, ρ is "bigger" than ρ_0).

In particular, if $s = 2$ then "big" means irred.

$s = 3$ then "big" means ρ is not the sum of 3 reps.

We need to assume ... characteristic of $R/m_R \geq n$.

Examples:

$\boxed{s=2}$ $\rho_0 = \rho_1 \oplus \rho_2$ $(\rho, R, I) \overset{\text{args used in MC}}{\rightsquigarrow}$ one can construct a lattice $\mathcal{L} \subset V_\rho = (\mathrm{Frac}(R))^n$

s.t $0 \to \rho_1 \otimes \mathcal{N} \to \mathcal{L}/I\mathcal{L} \to \rho_2 \otimes R/I \to 0 \rightsquigarrow$ gives elts in $H^1(F, \rho_1 \otimes \rho_2^\vee \otimes \mathcal{N})$

with $\mathcal{N}$ some torsion R-module s.t I dividing $\mathrm{Fitt}_R(\mathcal{N})$.

If ρ_1, ρ_2 and ρ are ordinary $\Rightarrow$ the image of

(would switch ρ_1, ρ_2 here if we wanted!)

$$\mathcal{N}^* \overset{\iota}{\hookrightarrow} H^1(F, \rho_1 \otimes \rho_2^\vee \otimes \mathcal{O}^\times)$$
$$\varphi \longmapsto \varphi \circ c_{\mathcal{N}}$$

lands in the ordinary class $\Rightarrow \mathrm{Im}(\iota) \subset \mathrm{Sel}^\Sigma(F, \rho_1 \otimes \rho_2^\vee \otimes \mathcal{O}^\times)$.

$\boxed{s=3}$ $\rho_0 = \rho_1 \oplus \rho_2 \oplus \rho_3$ (ρ, R, I)

can choose any of the reps, choose ρ_3.

We can construct a Galois stable lattice $\mathcal{L}$ s.t.

$\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ (not Galois stable) s.t.

$$0 \to N_1 \oplus N_2 \to \mathcal{L}/I\mathcal{L} \to \rho_3 \otimes R/I \to 0$$

$N_i = \mathcal{L}_i \otimes \rho_i/I$, N_i not Galois stable

$$\rho_{R/I} \simeq \begin{pmatrix} \rho_1 & B & D \\ C & \rho_2 & E \\ 0 & 0 & \rho_3 \end{pmatrix}$$

$BC \equiv 0 \pmod{I}$ either way get an extension..

If we have info about non existence of elements in

$Sel(\rho_i \otimes \rho_3^\vee)$ for $i = 1, 2$ $\Rightarrow$ get elts in other one.

It is difficult if one gets some extensions in each, so it is ideal if one knows that one cannot have one of the types of extensions ever.

Unitary Groups:

Fix $a \geq b \geq 0$, K = imag. quad. field.

$G_{a,b}$ = unitary group defined by the skew Hermitian matrix

$$T_{a,b} = \begin{pmatrix} & & -1_b \\ & \Theta & \\ 1_b & & \end{pmatrix}$$

Θ = skew-Hermitian and definite

$$G_{a,b}(\mathbb{R}) \simeq U(a,b) \supset U(a) \times U(b)$$

Consider a sequence $c_1 \geq c_2 \geq \cdots \geq c_b$

$c_{b+1} \geq \cdots \geq c_d$

$c_b \geq c_{b+1} + d \qquad d = a+b$

$$\vec{c} = (c_{b+1}, \ldots, c_d, c_1, \ldots, c_b)$$

$\rightsquigarrow$ defines an automorphy factor on the Hermitian domain

$$\mathfrak{h}_{a,b} = G_{a,b}(\mathbb{R}) / U(a) \times U(b)$$

Holomorphic automorphic form of weight $\vec{c}$.

- π automorphic rep. of $G_{a,b}(\mathbb{A})$

  $\pi = \pi_f \otimes \pi_\infty$ with π_∞ is a holomorphic discrete series of wt $\vec{c}$

- χ Hecke character of K

$$\chi_\infty(z) = \left(\frac{z}{\bar{z}}\right)^{\kappa} (z\bar{z})^{\kappa'} \qquad \kappa, \kappa' \in \tfrac{1}{2}\mathbb{Z}$$

$$2\kappa \equiv 2\kappa' \mod 2$$

- (Reg) $\quad c_b \geq \kappa + \frac{d}{2} - 1, \quad \kappa' - \frac{d}{2} - 1 \geq c_{b+1}$.

Given such an aut rep $\overset{\text{(conj.)}}{\rightsquigarrow}$ one has a Galois rep.

<u>Conjecture</u>: $\exists\ R_p(\pi) : G_K \longrightarrow GL_d(\overline{\mathbb{Q}}_p)$ s.t.

① $R_p(\pi)^\vee(1-d) \simeq R_p(\pi)^c \qquad c = \text{c.c.}$

② $L(R_p(\pi), s) = L(\pi^\vee, s + \frac{1-d}{2})$.

②$^{\text{weak}}$ $L^\Sigma(R_p(\pi), s) = L^\Sigma(\pi^\vee, s + \frac{1-d}{2}) \qquad \Sigma$ primes that ramify

We refer to this conjecture as $(\mathrm{Rep}_{a,b})$.

$G_{a+1,b+1}$ $\quad$ P parabolic stabilizing an isotropic line

$$P = MN \qquad M \simeq G_{a,b} \times \mathbb{G}_m/K$$

$\pi \times \chi$ rep of this Levi

$$\phi_s \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{a+1,b+1}(\mathbb{A})} (\pi \times \chi)\, \delta^s$$

modular char, $s \in \mathbb{C}$

We can then form the E.S.

$$E(\phi_s, s, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G_{a+1,b+1}(\mathbb{Q})} \phi_s(\gamma g) \qquad \mathrm{Re}(s) \gg 0, \ g \in G_{a+1,b+1}(\mathbb{A}).$$

$\chi' = \chi|_{\mathbb{A}_{\mathbb{Q}}^\times}$ $\quad$ Consider the two cases:

$$\begin{cases} \chi' = |\cdot|_{\mathbb{Q}}^{2\kappa'} \\ \chi' \neq |\cdot|_{\mathbb{Q}}^{2\kappa'} . \end{cases}$$

Under the second case, the evaluation at $s = \frac{1}{2} - \kappa'$ defines a holomorphic form on $G_{a+1,b+1}$. We denote this E.S. by

$$E(\pi, \chi, \phi) \qquad s = \tfrac{1}{2} - \kappa'$$

the holomorphic form of weight $(\kappa - \frac{d}{2} - 1, c_{b+1}, \ldots, c_d, c_1, \ldots, c_b, \kappa + \frac{d}{2} - 1)$

Moreover, the L-function of this automorphic form is given by

ϕ is not here b/c away from S it is unique!

$$L^S(E(\pi,\chi,\phi), s) = L^S(\pi, s)\, L^S(\chi, s - \kappa' + \tfrac{1}{2})\, L^S(\chi^{-c}, s + \kappa' + \tfrac{1}{2})$$

Therefore, the corresponding Galois rep (assuming $\mathrm{Rep}(a,b)$) is

$$R_p(E(\pi,\chi,\phi)) = R_p(\pi)(-1) \oplus \chi_p^c\, \varepsilon^{-\kappa' - d/2} \oplus \chi_p^{-1} \varepsilon^{\kappa' - d/2 - 1}.$$

We can choose Φ so that $E(\pi, \chi, \phi)$ is "ordinary at p".

(Remark: if π_p is ord. $\Rightarrow R_p(\pi)$ is ordinary)

We now study the Eisenstein ideal for this E.S. We deform the rep. $R(E(\pi,\chi))$. We can now try to perform the strategy used for the classical main conjecture.

We can do the lattice construction:

$$\begin{pmatrix} \chi_p^{-1}\varepsilon^{\kappa'-\frac{d}{2}-1} & C & D \\ B & R_p(\pi)(-1) & E \\ 0 & 0 & \chi_p^c\,\varepsilon^{-\kappa'-d/2} \end{pmatrix}$$

modulo the Eisenstein ideal.

This allows one to construct two type of cocycles. invariant under c.c

D gives element in $Sel_K(\underset{\substack{\uparrow \\ \text{Dir. cha.}\cdot\varepsilon^{-1}.}}{\chi_p'^{-1}}\,\varepsilon^{2\kappa'-1})^c = Sel_{\mathbb{Q}}(\chi_p'^{-1}\,\varepsilon^{2\kappa'-1})$

B gives element in $Sel_K(R_p(\pi)\otimes\chi_p^{-c}\,\varepsilon^{-\kappa'-\frac{d}{2}-1})$

So if we can show is trivial, then we get the Eisenstein ideal divides

In the $U(2,2)$ situation, we assume $Rep(2,2)$. Let $\mathbb{F}$ be a $GL(2)$ Hida family. $\Gamma_K = Gal(K_\infty/K)$, K_∞ $\mathbb{Z}_p^2$ max ext of K, p split in K. $\Lambda = \mathbb{I}[[\Gamma_K]]$ where $\mathbb{F}\in\mathbb{I}[[q]]$, $\mathbb{I}/\underset{\substack{\nwarrow \\ \text{weight variable of Hida family.}}}{\mathbb{Z}_p[[w]]}$ fin. ext., $\mathbb{I}$ integrally closed.

(*) $\rho_{\mathbb{F}} : G_{\mathbb{Q}} \to GL_2(\mathbb{I}) \Leftarrow \bar{\rho}_{\mathbb{F}}$ is abs. irred (assumption)

$$Sel_{K_\infty}(\rho_{\mathbb{F}}) = Sel_K(\rho_{\mathbb{F}} \otimes_{\mathbb{I}} \Lambda^*) \qquad \Lambda^* = Hom_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$$

$\mathcal{F}_{\mathbb{F},K}(s_+, s_-) \in \mathbb{I}[[s_+, s_-]] \simeq \Lambda$ char. power series of $X_{K_\infty}(\rho_{\mathbb{F}})$

under some tech. assump.

Thm A: (Skinner-U.) $\mathcal{L}^S_{\mathbb{F},K}((1+s_+)u^{-1}-1, s_-)\, \mathcal{L}^S_{\chi_{\mathbb{F}}^{-1}\omega}((1+s_+)^{-1}(1+w)^{-1}u^3-1) \mid$ Eisenstein ideal of the universal ordinary cuspidal Hecke alg for $U(2,2)$. $= Eis$

p-adic L-fctn

3 variable p-adic L-fctn interpolating two chars and 1 variable for weight

$\chi_{\mathbb{F}}$ = nebentypus for Hida family

under $U(2,2)$

Thm B: (Skinner-U.) If $\mathcal{L}_{\chi_{\mathbb{F}}^{-1}\omega}$ is a unit, then

$$Eis \mid \mathcal{F}_{\mathbb{F},K}((1+s_+)u^{-1}-1, s_-).$$

Cor: If $\mathbb{F}$ is the Hida family lifting the form of wt 2 attached to an ordinary elliptic curve then

$$\mathcal{L}^S_{\mathbb{F},K} \mid \mathcal{F}^S_{\mathbb{F},K}.$$

The case where $\chi' = |\cdot|^{2\kappa'}$:

$$R = R_p(\pi) \otimes \chi_p\, \varepsilon^{d/2-\kappa'}.$$

$$R^c \simeq R^\vee(1)$$

$$L(R,s) = \varepsilon(R,s)\, L(R,-s)$$

<u>Thm:</u> (Skinner-U) ① Assume $\mathrm{Rep}(a+1,b+1)$ (strong form, others only needed weak form)

(1) Assume $L(R,0)=0$.

then $\mathrm{rk}\ \mathrm{Sel}_K(R^\vee(1)) \geq 1$

② Assume $\mathrm{Rep}(a+2,b+2)$. If $L(R,s)$ vanishes at $s=0$ to an even order, then

$$\mathrm{rk}\ \mathrm{Sel}_K(R^\vee(1)) \geq 2.$$