Weinstein

3-22-11

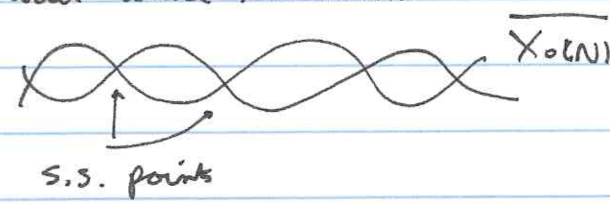
A semistable model for the tower of modular curves:

$X_n = X(p^n N)_{\overline{\mathbb{Q}}_p}$ $p \nmid N$

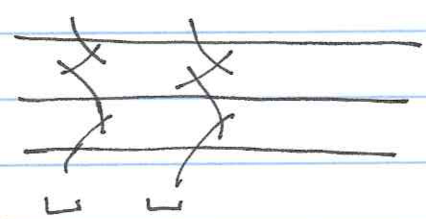
$\exists\ \mathfrak{X}_n / \breve{\overline{\mathbb{Z}}}_p$ flat, proper, $\mathfrak{X}_n \otimes \overline{\mathbb{F}}_p$ has only nodal singularities

The goal is to give a description of $\mathfrak{X}_n \otimes \overline{\mathbb{F}}_p$.

- $X_0(pN)$ has model whose reduction



- $X = X_0(p^2N)$ (Edixhoven '90)
- $X = X_0(p^3N)$ (Coleman-McMurdy '10)



s.s. reduction of deformation of s.s. elliptic curve E.

- $X(p^\infty) = \{(E, \alpha)/\sim : \alpha : \mathbb{Q}_p^2 \simeq V_p E\}$ ↺ $GL_2(\mathbb{Q}_p)$.

Lubin-Tate spaces:

Let $G_0/\overline{\mathbb{F}}_p$ be a p-div. group, formal, dim 1, ht $h>0$
($h=2$, $G_0 = E[p^\infty]$, $E/\overline{\mathbb{F}}_p$ s.s. ell. curve)

The Lubin-Tate space
$$\mathcal{M} = \varprojlim \mathcal{M}_K \qquad K \subseteq GL_h \mathbb{Q}_p$$

A point of $\mathcal{M}(\mathbb{C}_p)$ is a triple $(G, \iota, \alpha)/\sim$

- G = p-div 1-dim formal group $/\mathcal{O}_{\mathbb{C}_p}$
- $\iota: G_0 \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \longrightarrow G \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ quasi-isogeny
- $\alpha: \mathbb{Q}_p^h \xrightarrow{\sim} V_pG = T_pG \otimes \mathbb{Q}_p$

Then $\mathcal{M} \circlearrowleft GL_h(\mathbb{Q}_p) \times J$ $\quad B = \mathrm{End}^\circ G_0/\mathbb{Q}_p$ div. alg.
$J = B^\times$

__Thm (Harris-Taylor):__ $H_c^*(\mathcal{M}, \overline{\mathbb{Q}}_\ell)$ realizes the L.L.C. and J.L.C. for $GL_h(\mathbb{Q}_p)$ (supercuspidal part).

__Goal:__ Find formal model $\hat{\mathcal{M}}$ with semistable reduction.

The Gross-Hopkins Period Map

__Thm (GH, '94):__ There exists a surjective étale map

$$\mathcal{M} \longrightarrow \mathbb{P}^{h-1}(\mathbb{C}_p)$$

with Galois group $GL_h(\mathbb{Q}_p)$.

Sketch:
Given G_0 have Dieudonné module $D(G_0)/W(\overline{\mathbb{F}}_p)$ free rk h, if G is a lift of G_0 have "Hodge sequence"

$$0 \to \omega_G \longrightarrow D(G/\mathcal{O}_{\mathbb{C}_p}) \longrightarrow \mathrm{Lie}(G^\vee) \to 0$$

ι induces an isomorphism

$$\begin{array}{ccc} D(G_0)\otimes \mathbb{C}_p & \longrightarrow & D(G/\mathcal{O}_{\mathbb{C}_p})\otimes \mathbb{C}_p \\ \cup\!\mid & & \cup\!\mid \\ \mathrm{Fil} & \longrightarrow & \omega_G \otimes \mathbb{C}_p \end{array}$$

$\mathrm{Fil} \in \mathbb{P}^{h-1}(\mathbb{C}_p)$.

$(G,\iota) \longrightarrow \mathrm{Fil} \in \mathbb{P}^{h-1}(\mathbb{C}_p)$.

<u>Finite Dimensional Vector Spaces:</u> (e.g. Banach-Colmez spaces)

- A_{cris} = p-adically complete $W(\overline{\mathbb{F}}_p)$-alg. (flat)
- $\Theta : A_{cris} \longrightarrow \mathcal{O}_{\mathbb{C}_p}$ and $\ker\Theta$ is a nilp. PD ideal
  ($x \in \ker\Theta$, $x^n/n! \in A_{cris}$)
- A_{cris} is universal for these properties
- $\varphi : A_{cris} \to A_{cris}$ is $W(\overline{\mathbb{F}}_p)$ semilinear homom.

Let $B^+_{cris} = A_{cris}[\frac{1}{p}]$.

<u>Fact:</u> $(B^+_{cris})^{\varphi=1} = \mathbb{Q}_p$.

$U = (B^+_{cris})^{\varphi=p} / \mathbb{Q}_p$ $(= U_1)$

$= \{ (x_0, x_1, \dots) : x_n \in \mathcal{O}_{\mathbb{C}_p},\ |x_n - 1| < 1,\ x_n^p = x_{n-1} \}$

$(x_n) + (y_n) = (x_n y_n)$

$a \in \mathbb{Z}_p$ $\quad a(x_n) = (x_n^a)$ $\qquad \frac{1}{p}(x_n) = (x_{n+1})$

$\theta((x_n)) = \log x_0$

$t = (1, \zeta_p, \zeta_{p^2}, \dots) \in \ker\theta = \mathbb{Q}_p t$ $t =$ "$2\pi i$" period of $\mathbb{G}_m$.

$0 \to \mathbb{Q}_p t \to U \xrightarrow{\theta} \mathbb{C}_p \to 0$ dimension $(1,1)$ in Colmez's terminology.

$U_h = (B_{cris}^+)^{\varphi^h = p}$ $\mathbb{Q}_{p^h}/\mathbb{Q}_p$

$0 \to \mathbb{Q}_{p^h} t_h \to U_h \xrightarrow{\theta} \mathbb{C}_p \to 0$

$t_h =$ period of LT formal dim 1 $\mathbb{Z}_{p^h}$-module

$\mathbb{Q}_{p^h} := W(\mathbb{F}_{p^h})[\frac{1}{p}]$

U

$\cup$

$U^0 = \{(x_n) : 0 < v(x_0 - 1) < \frac{p}{p-1}\}$

↑ this forces $v(x_n - 1) < \frac{1}{p^{n-1}(p-1)}$.

$\hat{U}^0 = \mathrm{Spf} \varinjlim \mathcal{O}_{\mathbb{F}_p}[[y_n, z_n]]$ $y_n z_n = p^{\frac{1}{p^{n-1}(p-1)}}$.

$\hat{U}^0 \otimes \overline{\mathbb{F}}_p = \mathrm{Spec}\left(\frac{\overline{\mathbb{F}}_p[[y,z]]}{yz}\right)^{perf.}$

Cry

Cristalline period maps (Fargues, Faltings, Fontaine)

(G, ι, α)

$G =$ p-div. grp.

$\iota : G_0 \otimes \mathcal{O}/p \longrightarrow G \otimes \mathcal{O}/p$

$\alpha : \mathbb{Q}_p^h \xrightarrow{\sim} V_p G$

$$\left[\begin{array}{l} A/\mathbb{C} \quad \text{ab. var.} \\ H_1(A,\mathbb{Q}) \longrightarrow H^1_{dR}(A/\mathbb{C})^* \\ \quad \Downarrow \\ H_1(A,\mathbb{Q}) \otimes \mathbb{Q}_p = T_p A \end{array}\right]$$

$$\begin{array}{ccc} V_p G & \xrightarrow{\sim} & D(G)(A_{cris} \to \mathcal{O}_{\mathbb{C}_p})[\tfrac{1}{p}]^{\varphi = p} \\ \alpha^{-1} \downarrow & & \downarrow \iota_* \\ \mathbb{Q}_p^h & \longrightarrow & (D(G_0) \otimes B^+_{cris})^{\varphi = p} \simeq (B^+_{cris})^{\varphi^h = p} \\ & & = U^h \end{array}$$

$$\begin{array}{ccccc} \mathcal{M}_{LT} & \longleftrightarrow & U_h^{\oplus h} & & (u_1, \ldots, u_h) \\ \downarrow & \text{cartesian} & \downarrow & & \downarrow \\ Mat_h(\mathbb{C}_p)^{rk = h-1} & \longrightarrow & Mat_h(\mathbb{C}_p) & & \theta(\varphi^i(u_j)) \quad 0 \le i \le h-1, \; 1 \le j \le h \\ \downarrow & & & & \\ \mathbb{P}^{h-1} & & & & \end{array}$$

Aside: $h = 2 \qquad (v_1, v_2)$

$$\varphi(v_1) v_2 - v_1 \varphi(v_2) \in \varepsilon U^2 \cap \ker\theta \qquad \varphi(\varepsilon) = -\varepsilon$$

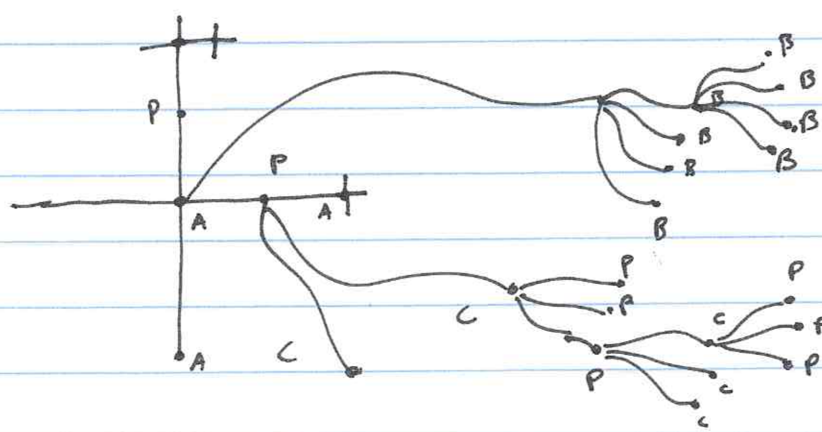
h=2:

$\mathcal{M}^{cm} \subseteq \mathcal{M}$

Thm: $\mathcal{M} \setminus \mathcal{M}^{cM}$ has a formal model $\hat{\mathcal{M}}/\mathcal{O}_{\mathbb{C}_p}$ s.t.

- $\hat{\mathcal{M}} \otimes \overline{\mathbb{F}}_p$ has singularities w/ local rings
$$\left(\mathbb{F}_p[[x,y]]/_{xy} \right)^{perf}$$
- $\hat{\mathcal{M}} \otimes \overline{\mathbb{F}}_p$ has irred. components of the form C^{perp} where $C/\overline{\mathbb{F}}_p$ is proj. smooth curve of the type:
  - A • $xy^p - x^p y = 1$ (p odd)
  - B • $y^p - y = x^{p+1}$
  - C • $y^p - y = x^2$
  - P • $\mathbb{P}^1$

Let Γ = dual graph of $\hat{\mathcal{M}} \otimes \overline{\mathbb{F}}_p$. vertices $v \in \Gamma \longleftrightarrow$ components C_v.



(p=3 case)

$\circlearrowleft$
$GL_2(\mathbb{Q}_p)$

Weinstein
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p97

$SL_2(\mathbb{Z}_p) \to \mathrm{Aut}\, A$

$\Gamma \longrightarrow \mathrm{Aut}\, B$