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Construction of eigenvarieties and coherent cohomology:

(w/ Andreatta, Iovita, Stevens, Stroh)

I. The program

K # field

$X/_K$ a PEL Shimura variety

Hecke $\circlearrowright$ $H^0(X,$ coherent. auto sheaf$)$

Ex: X = compactified modular curve

$k \in \mathbb{Z}$, $H^0(X, \omega^k)$

a) p-adically interpolate the automorphic coherent sheaves restricted to an open rigid subspace of X_{rig}.

→ define overconvergent modular forms of any p-adic weights.

→ families

b) Coleman's spectral theory to get eigenfamilies.

c) Prove a "small slope forms are classical theorem"

Remark: 1) This is the coherent counterpart to the cohomological construction

Hecke $\circlearrowright$ $H^{\bullet}_{Betti}(X_{\mathbb{C}},$ some level system$)$

These two constructions should be related by overconvergent Eichler–Shimura (see next talk)

2) In the ordinary case, Hida did this using p-adic modular forms.

3) On modular curves, a) gives a new way to look at Coleman's original construction.

II. The Key Idea in the modular curve case:

a) Notation

$X/_{\mathbb{Z}_p}$ compactified modular curve of level $\Gamma_1(N)$, $(N,p)=1$

$\mathcal{E} \longrightarrow X$ semi abelian scheme, with section e

$e^*(\Omega^1_{\mathcal{E}/X}) = \omega_{\mathcal{E}}$

$$\begin{tikzcd} \tau^\times \arrow[r, hook] \arrow[dr, "\pi"'] & \tau \arrow[d] \\ & X \end{tikzcd}$$

$\mathbb{G}_m$ acts on $\tau^\times$

$\tau^\times = \mathrm{Isom}(\mathcal{O}_X, \omega_{\mathcal{E}})$

$\tau = \mathrm{Hom}(\mathcal{O}_X, \omega_{\mathcal{E}})$

$\pi_* \mathcal{O}_{\tau^\times}[k] = \omega^k$ wt k modular forms

$k \in \mathbb{Z}$ action of $\mathbb{G}_m$

b) p-adic interpolation:

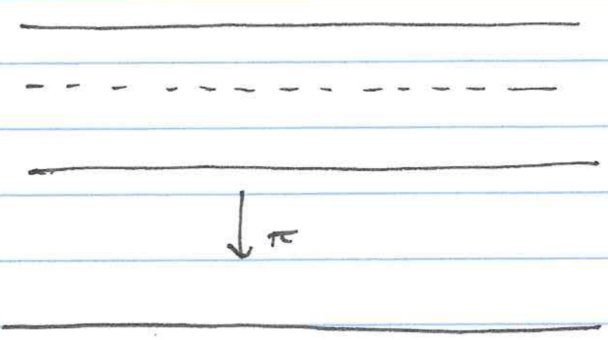
Let $W = \mathrm{Hom}_{gr}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ we want $\omega^k, \forall k \in W$

$\mathbb{Z} \hookrightarrow W$

X_{rig} rigid space assoc. to X

$\mathcal{T}_{rig}$

$\mathcal{T}^\times_{rig} = \mathcal{T}_{rig} - \{0\}$



$\circlearrowleft \mathbb{Z}_p^\times$

$\mathcal{T}^\times_{rig} \subset \mathcal{T}_{rig}$ $\{E/\mathcal{O}_K, \omega \in \omega_E\}$

$\downarrow \pi$

X_{rig} $\{E/\mathcal{O}_K\}$

$$\pi_* \mathcal{O}_{\mathcal{T}^\times_{rig}}[k] = \omega^k$$

$$\circlearrowleft \mathbb{Z}_p^\times$$

<u>Naive idea:</u> $\forall\, k \in W$, $\omega^k = \pi_* \mathcal{O}_{\mathcal{T}^\times_{rig}}[k]$, but this is 0!

$$\{\varphi : B(0,1) - \{0\} \longrightarrow \mathbb{C}_p, \text{ analytic}, \varphi(\lambda^{-1} z) = \kappa(\lambda)\varphi(z), \lambda \in \mathbb{Z}_p^\times\} = 0$$

if $\kappa \notin \mathbb{Z}$.

$$\begin{array}{ccc} \kappa : \mathbb{Z}_p^\times & \longrightarrow & \mathbb{C}_p^\times \\ \tilde{\kappa} \searrow & & \uparrow \\ & \mathbb{Z}_p^\times(1 + p^w \mathcal{O}_{\mathbb{C}_p}) & \end{array} \qquad \exists\, w > 0$$

$\Rightarrow \{\varphi : \mathbb{Z}_p^\times(1 + p^w \mathcal{O}_{\mathbb{C}_p}) \longrightarrow \mathbb{C}_p, \varphi(\lambda^{-1} z) = \kappa(\lambda)\varphi(z)\} = \mathbb{C}_p \tilde{\kappa}$.

We need to cut a subspace in $\mathcal{T}^\times_{rig}$.

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c) The Hodge-Tate map:

$G/_S$ finite flat group scheme ω_G = canonical sheaf

$$HT : G^D \longrightarrow \omega_G$$
$$\phi : G_T \to \cdot \quad \rightsquigarrow \phi^* \frac{dt}{t}$$

c2) Canonical subgroup:

R is a normal admissible $\mathcal{O}_K$-algebra.
Let $A \to S = \operatorname{Spec} R$ g-dim abelian scheme

$$v : \mathcal{O}_{\mathbb{C}_p} \longrightarrow \mathbb{R} \cup \{\infty\} \quad v(p) = 1$$

$$v : \mathcal{O}_{\mathbb{C}_p}/p\,\mathcal{O}_{\mathbb{C}_p} \longrightarrow [0,1]$$

$$Hdg : S_{rig} \longrightarrow [0,1]$$
$$x : \operatorname{Spec} \mathcal{O}_L \hookrightarrow \operatorname{Spec}(R) \longrightarrow v(\operatorname{Hasse}(x^* A / \mathcal{O}_L/p\mathcal{O}_L))$$

Thm: Let $n \in \mathbb{N}$, $\sup_{x \in S_{ris}} Hdg(x) \leq \frac{p-1}{p(p^n-1)}$

$\exists$ a "canonical subgroup" $H_n \subseteq A[p^n]|_{S_{ris}}$ which can be characterized as follows: $\left(H_n \overset{\text{locally isom}}{\cong} (\mathbb{Z}/p^n\mathbb{Z})^g\right)$

Let R' be an admissible normal $\mathcal{O}_K$-algebra $R' \leftarrow R$ is finite, $R'_K \leftarrow R_K$ étale and $A[p^n]|_{R'_K}$ is trivial.

We have an exact sequence

$$0 \longrightarrow H_n(R'_K) \longrightarrow A[p^n](R') \xrightarrow{HT} \omega_{A[p^n]} \otimes_R R'$$

$\forall x \in S_{rig} \quad v(\det(HT \otimes 1(x))) = \frac{Hdg(x)}{p-1}$

<u>Remark:</u> Suppose A is p-polarized at p.

$H_n^\perp = H_n \quad HT : H_n^D(R') \longrightarrow \omega_{A[p^n]} \otimes_R R'$.

$H_n^D = H_n$ dual

<u>(3) Ball fibration:</u>

For all $v \in [0,1]$ $\quad X(v) = \{x \in X_{rig},\ Hdg(x) \le v\}$.

<u>Thm:</u> $\forall n \in \mathbb{N}$, we have a diagram

$$\begin{tikzcd}
 & \mathcal{F}_n \arrow[r, hook] \arrow[dl] \arrow[d, "\pi"] & \tau_{rig}^{\times}\big|_{X\left(\frac{p-1}{p(p^n-1)}\right)} \arrow[dl] \\
(H_n^D)^{\times} \arrow[r] & X\left(\frac{p-1}{p(p^n-1)}\right) &
\end{tikzcd}$$

$(H_n^D)^\times = \mathrm{Isom}(\mathbb{Z}/p^n\mathbb{Z}, H_n^D)$

$\mathcal{F}_n = \left\{ E/\mathcal{O}_K,\ \begin{array}{l} P \in H_n^D(K) \text{ of order } p^n,\ \omega \in \omega_E \\ Hdg(E) \le \frac{p-1}{p(p^n-1)} \quad HT(P) = \omega|_{\omega_{E[p^n]}} \end{array} \right\}$.

$\mathcal{F}_n$ is locally union of $(p-1)p^{n-1}$ balls of radius $\simeq p^{-n}$.

<u>Def:</u> Let $\kappa \in W$, if $n \gg 1$,

$\pi_* \mathcal{O}_{\mathcal{F}_n}[\kappa] = \omega^{\kappa}$

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$M_\kappa^\dagger = \lim_{v\to 0} H^0(X(v), \omega^\kappa)$ overconvergent mod forms of weight κ.

ω^κ varies analytically with κ.

<u>(4) Integral structure modular def</u>

An integral weight κ, v small enough for κ, overconvergent modular form is a functorial rule if

$$(E/_R, \omega, P) \rightsquigarrow f(E/_R, \omega, P) \in R$$

- R is normal admissible
- $v_p(\mathrm{Hdg}) \leq v$
- $\omega \in W_{E/R}$, $P \in H_n^D(R)$ is of order p^n, $HT(P) = \omega|_{W_{E[p^n]}}$

i.e., $f(E/_R, \lambda^{-1}\omega, \lambda^{-1}P) = \kappa(\lambda) f(E/_R, \omega, P)$

<u>III Generalizations:</u>

Siegel variety.

$X/_{\mathbb{Z}_p}$ (toroidal compactification) of moduli of $A, H_1 \subset \cdots \subset H_g \subset A[p], \psi_N$

A is a g-dim. prin. polarized abelian scheme

$H_\cdot$ a totally isotropic flag.

$\mathcal{G} \underset{e}{\rightleftarrows} X$ semi-abelian scheme

$\omega_{\mathcal{G}} = e^* \Omega^1_{\mathcal{G}/X}$

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$$\tau^\times = \mathrm{Isom}(\mathcal{O}_X^g, \omega_{cy}) \curvearrowleft K_g$$
$$\downarrow \pi$$
$$X$$

(can't read this part)

$\forall \kappa \in X(\tau)^*$

$\omega^\kappa = \pi_* \mathcal{O}_{\tau^\times}[\kappa]$ action of B

<u>Remark:</u> ω^κ is locally free, but its rank depends on κ

$\longleftrightarrow \mathrm{Ind}_B^{GL_g} \kappa.$

<u>Thm:</u> $W = \mathrm{Hom}_{gr}(T(\mathbb{Z}_p), \mathbb{C}_p^*)$

$\forall \kappa \in W$, $\exists v, w \in \mathbb{Q}_{>0}$, v small enough, w big enough

and a Banach sheaf

$$\omega_w^{+\kappa}\Big|_{X(v)} \underset{\substack{\text{locally}\\ \text{étale}}}{\simeq} p^{-w}\text{-analytic } \mathrm{Ind}_{B(\mathbb{Z}_p)}^{\text{Iwahori}} \kappa.$$

$$\lim_{\substack{v\to 0\\ w\to\infty}} H^0(X(v), \omega_w^{+\kappa}) = \text{space of locally analytic overconvergent forms}$$

If $\kappa \in X^+(T)$ there is an exact sequence

$$0 \to \omega^\kappa|_{X(v)} \longrightarrow \omega_w^{+\kappa} \longrightarrow \bigoplus_\alpha \omega_w^{+\alpha\kappa}$$
$$\wr|$$
$$\mathrm{Ind}_B^{GL_g} \kappa \longleftrightarrow \mathrm{an\,Ind}_B^{IW}$$

· $\omega_w^{+\kappa}$ vary analytically with κ.

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$f \in H^0_{cusp}(X, \omega^\kappa)$ eigen. for $\mathcal{H}^{Np} \otimes \mathbb{Z}_p$.

$\mathcal{H}^{Np}$ unramified Hecke alg. outside Np.

$\mathcal{U}_p = \mathbb{Z}_p[U_{p,1}, \dots, U_{p,g}]$ Hecke alg at p.

$\Theta_f : \mathcal{H}^{Np} \otimes \mathcal{U}_p \longrightarrow \mathbb{C}_p$

<u>Thm:</u> $\exists\, \mathcal{U} \underset{open}{\subset} W$ $\quad w : \mathcal{X}_f \longrightarrow \mathcal{U}$, $x_f \longmapsto \kappa$ ($x_f \in \mathcal{X}_f$)

where

- w is finite, inj, x_f is ? of dim ?
- $\Theta : \mathcal{H}^{Np} \otimes \mathcal{U}_p \longrightarrow \mathcal{O}_{\mathcal{X}_f}$.
- $\Theta|_{x_f} = \Theta_f$

$\forall\, \kappa' = (\kappa'_1, \dots, \kappa'_g) \in X(T)^+ \cap \mathcal{U}$ s.t.

- $\kappa'_g - \frac{g(g+1)}{2} > v(\theta_f(U_{p,g}))$

- $\kappa'_i - \kappa'_{i+1} + 1 > v(\theta_f(U_{p,i}))$ $\quad 1 \le i \le g-1$

small slope $\Rightarrow$ classical thm.

$g \in W^{-1}(\kappa')$ then Θ_f comes from

a form ~~in~~ in $H^0_{cusp}(X, \omega^{\kappa'})$