An overconvergent Eichler-Shimura map

Let $p \geq 5$, $N \geq 3$, $p \nmid N$, K = finite ext. of $\mathbb{Q}_p$,
$\bar{K}$ an alg. closure of K, $G_K = \mathrm{Gal}(\bar{K}/K)$.

Let $k \in W(K) = \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, K^\times) \supseteq \mathbb{Z}$
$$(t \mapsto t^k) \leftarrow\!\shortmid k$$

Overconvergent modular symbols of wt k: $T_0 = \mathbb{Z}_p^\times \times \mathbb{Z}_p$
$I_W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) : p \mid c \right\}$ acts on T_0. T_0 also has diagonal action of $\mathbb{Z}_p^\times$.

$$A_k = \{ f : T_0 \to K \mid \text{analytic}, \ f(tx) = k(t) f(x), \ t \in \mathbb{Z}_p^\times, x \in T_0 \} \circlearrowleft I_W$$

$$D_k = \mathrm{Hom}_{cts, K}(A_k, K) \circlearrowleft I_W \supseteq \Gamma = \Gamma_1(N) \cap \Gamma_0(p)$$

(analytic distributions on T_0 of weight k)

- $H^1(\Gamma, D_k)$ = overconvergent modular symbols of weight k.
This space has a natural action of T_ℓ, $\ell \nmid Np$, U_p.

- $H^1(\Gamma, D_k) = H^1_{et}(Y(N,p)_{\bar{K}}, D_k) \circlearrowleft G_K$
Thus, $H^1(\Gamma, D_k)$ is a Hecke module with an action of G_K.

In the talk, $H^1(\Gamma, D_k)$ will be discussed as a G_K-rep. from a p-adic Hodge th. point of view

Outline: Let $w \in \mathbb{Q}$, $0 < w < \frac{p}{p+1}$

$$\begin{tikzcd} X(N,p) \ni (E, \psi_N, C) \arrow[d] & & X(N,p)(w) \arrow[d, "\simeq"] \\ X_1(N)/K \supseteq X_1(N)(w) \arrow[u] & (E, \psi_N) \arrow[ul, mapsto] & \end{tikzcd}$$

$$X_1(N)/K \supseteq X_1(N)(w) = \{ x \in X_1(N) : |E_{p-1}(x)| \geq p^{-w} \}$$

affinoid over K

Let $\underline{\omega}_K^{\kappa}$ be an invertible sheaf on $X(N,p)(w)$ s.t.
$H^0(X(N,p)(w), \underline{\omega}_K^{\kappa})$ = are overconvergent mod. forms of weight κ. ($\tau : \mathbb{F}_p^{\times} \to \mathbb{Z}_p^{\times}$ Teich. char., $s \in \mathcal{O}_K$,
$\kappa : \mathbb{Z}_p^{\times} \to K^{\times}$ $\quad \kappa(t) = \tau(t)\langle t \rangle^s$, $0 \leq t \leq p-1$.)

1) Define a G_K-equiv., Hecke equiv. map

$$H^1(\Gamma, D_\kappa) \hat{\otimes}_K \mathbb{C}_p \longrightarrow H^0(X(N,p)(w), \underline{\omega}_K^{\kappa+2}) \hat{\otimes}_K \mathbb{C}_p$$

2) If $\kappa = k \geq 0$, $k \in \mathbb{Z}$, if $0 \leq h < k+1$, then we can prove

$$H^1(\Gamma, D_k)^{(\leq h)} \hat{\otimes} \mathbb{C}_p \longrightarrow H^0(X(N,p)(w), \underline{\omega}^{\kappa+2})^{(\leq h)} \hat{\otimes} \mathbb{C}_p$$

is surj.

Consequence: $k \in \mathbb{Z}_{\geq 0}$, There is a map

$$D_k \longrightarrow V_k = \mathrm{Sym}^k(K^2).$$

This induces a map

$$H^1(\Gamma, D_k)^{(\leq h)} \xrightarrow{\simeq} H^1(\Gamma, V_k)^{(\leq h)}.$$

$$\begin{tikzcd} H^1(\Gamma, D_k)^{(\leq h)} \otimes \mathbb{C}_p \arrow[r] \arrow[d, phantom, "\wr"] & H^0(X(N,p)(w), \underline{\omega}^{\kappa+2})^{(\leq h)} \otimes \mathbb{C}_p \\ H^1(\Gamma, V_k)^{(\leq h)} \otimes \mathbb{C}_p \arrow[r, two heads] & H^0(X(N,\cancel{p})\cancel{(w)}, \omega^{\kappa+2})^{(\leq h)} \otimes \mathbb{C}_p \arrow[u, hook] \end{tikzcd}$$

- 1) $H^1(\Gamma, D_k)\hat{\otimes}\mathbb{C}_p \xrightarrow{\varphi} H^0(X(N,p)(w), \underline{\omega}^{k+2})\hat{\otimes}\mathbb{C}_p$

Eichler-Shimura map.

2) $k \in \mathbb{Z}_{\geq 0}$, $0 < h < k+2$ $\quad\varphi$ is surj.

1)

$$\begin{array}{ccc} & X(N,p)(w) & \subseteq X(N,p) \\ & \wr\| & \downarrow \\ \mathcal{X} \hookrightarrow & X_1(N)(w) & X_1(N) \end{array}$$

$\mathcal{X}$ = p-adic compl. of open consisting of the complement of ∞ in the exc div. of the blow-up of $X_1(N)_{\mathcal{O}_K}$ at (p^w, E_{p-1})

$\mathcal{X}_K = X_1(N)(w) \cong X(N,p)(w) = X$

$\mathfrak{X}$ = Falting's site assoc. to $(\mathcal{X}, X)$.

- category of open sets: $E_{(\mathcal{X},X)}$: objects $(\mathcal{U}, W)$,

$\mathcal{U} \to \mathcal{X}$ étale (Kummer-log-étale)

$W \to \mathcal{U}_{\bar{K}}$ finite-étale (Kummer log étale)

morphisms: ~~[illegible]~~

$(f,g): (\mathcal{U}, W) \to (\mathcal{U}', W')$

$f: \mathcal{U} \to \mathcal{U}'$ over $\mathcal{X}$, $\quad g: W \to W'$ with $\begin{array}{ccc} W & \xrightarrow{g} & W' \\ \downarrow & & \downarrow \\ \mathcal{U}_{\bar{K}} & \xrightarrow{f_{\bar{K}}} & \mathcal{U}'_{\bar{K}} \end{array}$

$\{(\mathcal{U}_i, W_i) \to (\mathcal{U}, W)\}_{i\in I}$ is of type (α) if $\{\mathcal{U}_i \to \mathcal{U}\}_{i\in I}$ is a covering and

$W_i \cong W \times_{\mathcal{U}} \mathcal{U}_i \quad \forall i$

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and of type (β) if $\exists\, U$ s.t. $U_i \cong U$ $\forall i \in I$ and $\{W_i \to W\}_{i \in I}$ is a covering.

$$\mathcal{X}^{ket} \xrightarrow{v} \mathfrak{X} \xrightarrow{u} X_{\bar{K}}^{fket}$$

$$(U, W) \longmapsto W$$

$$U \longmapsto (U, U_{\bar{K}}).$$

$\underline{\omega}^{k}$, $v^*(\underline{\omega}^{k}) \in Sh(\mathfrak{X})$

$v^*(\mathcal{O}_{\mathcal{X}})$-module

D_k Banach module ($D(\mathbb{Z}_p^\times, K)$)

$D_k^\circ = \{\mu : \|\mu\| \le 1\}$, $\{D_k/p^n D_k^\circ\}_{n \geq 0}$.

$u^*(D_k^\circ)$ = cont. sheaf on $\mathfrak{X}$.

$\mathcal{O}_{\mathfrak{X}}(U, W)$ = normalization of $\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ in $\mathcal{O}_W(W)$.

$\hat{\mathcal{O}}_{\mathfrak{X}} = \{\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}}\}_{n \geq 0}$.

- $D_{k,\mathfrak{X}} = \{D_k^\circ/p^n D_k^\circ \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{X}}/p^n \mathcal{O}_{\mathfrak{X}}^{(1)}\}_{n \geq 0}$ sheaf on $\mathfrak{X}$.
- $\underline{\omega}_{\mathfrak{X}}^{k} = \{v^*(\underline{\omega}^{k}) \otimes_{v^*(\mathcal{O}_{\mathcal{X}})} \mathcal{O}_{\mathfrak{X}}/p^n \mathcal{O}_{\mathfrak{X}}\}_{n \geq 0}$

<u>Thm:</u> 1) There is a nat. G_K-equiv. map

$$H^1(\mathfrak{X}, D_{\kappa,\mathfrak{X}})[\tfrac{1}{p}] \longrightarrow H^0(\mathfrak{X}, \underline{\omega}_{\mathfrak{X}}^{\kappa+2})[\tfrac{1}{p}]$$

$$\wr\wr$$

$$H^0(X(N,p)(w), \underline{\omega}^{\kappa+2})\,\hat{\otimes}\,\mathbb{C}_p$$

2) Both terms have actions of Hecke operators T_ℓ, $\ell \nmid Np$, U_p s.t. if $h<\infty$,

$$\left(H^1(\mathfrak{X}, D_{\kappa,\mathfrak{X}})[\tfrac{1}{p}]\right)^{(\le h)} \xrightarrow{\sim} H^0(X(N,p)(w), \underline{\omega}^{\kappa+2})^{(\le h)} \otimes \mathbb{C}_p$$

$$\begin{tikzcd}
H^1(\Gamma, D_\kappa)\hat{\otimes}\mathbb{C}_p \arrow[r] & H^1(\mathfrak{X}, u_*(D_\kappa))\hat{\otimes}\mathbb{C}_p \arrow[r] & H^1(\mathfrak{X}, D_{\kappa,\mathfrak{X}})[\tfrac{1}{p}] \arrow[d] \\
& & H^0(X(N,p)(w), \underline{\omega}_\kappa^{\kappa+2})\hat{\otimes}_K\mathbb{C}_p
\end{tikzcd}$$

(This gives the 1st map above.)

<u>On the proof of theorem:</u>

D_κ : $D_{\kappa,\mathfrak{X}}$ has a filtration:

$$D_{\kappa,\mathfrak{X}} = \mathrm{Fil}^{-1} \supseteq \mathrm{Fil}^0 \supseteq \mathrm{Fil}^0 \supseteq \cdots$$

- $\mathrm{Fil}^m/\mathrm{Fil}^{m+1} \cong \underline{\omega}_{\mathfrak{X}}^{\kappa-2m-2}(m+1)$, $m \geq -1$.

<u>Lemma:</u> 1) $H^0(\mathfrak{X}, D_{\kappa,\mathfrak{X}}/\mathrm{Fil}^m) = H^0(\mathcal{X}, \underline{\omega}^{\kappa-2m+4})\hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\mathbb{C}_p}(m-1)$

2) $H^1(\mathfrak{X}, D_{\kappa,\mathfrak{X}}/\mathrm{Fil}^m)[\tfrac{1}{p}] = H^0(X, \underline{\omega}^{\kappa+2})\hat{\otimes}_K\mathbb{C}_p$

3) $H^i(\mathfrak{X}, D_{\kappa,\mathfrak{X}}/Fil^m)[\frac{1}{p}] = 0 \quad \forall i \geq 2.$

$h \geq$, $j \in \mathbb{Z}$ $j > h$

$$0 \to Fil^j \to D_{\kappa,\mathfrak{X}} \to D_{\kappa,\mathfrak{X}}/Fil^j \to 0$$

$$H^1(\mathfrak{X}, Fil^j)[\tfrac{1}{p}] \to H^1(\mathfrak{X}, D_{\kappa,\mathfrak{X}})[\tfrac{1}{p}] \to H^1(\mathfrak{X}, D_{\kappa,\mathfrak{X}}/Fil^j)[\tfrac{1}{p}]$$

$$\shortparallel$$

$$H^0(X, \underline{\omega}^{\kappa+2}) \hat{\otimes}_K \mathbb{C}_p$$