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Local-global compatibility and monodromy

I. Intro:

L CM field, $\iota_\ell : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, ℓ prime

Π cusp. auto. rep. of $GL_n(\mathbb{A}_L)$ s.t.

- $\Pi^\vee \simeq \Pi \circ c$
- reg. alg.

$R_\ell(\Pi) : Gal(\bar{L}/L) \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell)$ be assoc. to Π.

Thm: Let y be a place of L above $p \neq \ell$, Then

$$WD\left(R_\ell(\pi)|_{Gal(\bar{L}_y/L_y)} \right)^{F\text{-s.s.}} \simeq \underbrace{\iota_\ell^{-1} \mathcal{L}_{n,y}}_{\text{Local Langlands}}(\Pi_y).$$

WD-rep $(V, \underbrace{r}_{\text{rep. of } W_K}, \underbrace{N}_{\text{nilp. end.}})$

K - p-adic field

Thm known for: (previously known)

- n odd
- n ~~odd~~ even, Π sat. regularity cond. (Shin-regular)
- Π sq. integrable at a finite place

In general, up to semisimplification (loses trace of N)

We will extend this to Fr-s.s. by identifying monodromy operators.

$\mathcal{O}_K$ r.o.i, ϖ = unif, $\mathcal{O}_K/\varpi = \mathbb{F}$, $\#\mathbb{F} = q$

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<u>Def</u>: (V, r, N) is <u>strictly pure</u> of weight k, $k \in \mathbb{R}$, if for some lift ϕ of Frob_{v_K} every eigenvalue α of $r(\phi)$ is a Weil q^k-number.

X proper smooth $/\mathbb{F}$, $WD(H^i(X \times_{\mathbb{F}} \overline{\mathbb{F}}, \overline{\mathbb{Q}}_\ell))$ is strictly pure of weight i. $(N = 0)$

Mixed if it has an inner filt. s.t. the i^{th} graded piece is strictly pure of weight i.

$$N(\mathrm{Fil}_i V) \subset \mathrm{Fil}_{i-2}(V).$$

pure of wt k, mixed all wts in $k + \mathbb{Z}$

$i > 0$, $N^i : gr_{k+i} V \xrightarrow{\sim} gr_{k-i} V$

<u>Conj</u>: X smth proper $/K$,

$WD(H^i(X \times_K \overline{K}, \overline{\mathbb{Q}}_p))$ is pure of weight i.

<u>Properties</u>: 1) purity preserved under fin. ext. L/K

2) rec is pure $\Leftrightarrow$ $\sigma\pi$ tempered $\forall \sigma \in \mathrm{Aut}(\mathbb{C})$

3) given (V, r) with r semi-simple, $\exists$ at most one choice of N making (V, r, N) pure.

From 2) and 3), enough to prove

(A) $WD(R_\ell(\pi)|_{\mathrm{Gal}(\overline{L}_y/L_y)})^{F\text{-s.s.}}$ is pure

(B) $\left.\begin{array}{l}\Pi \text{ conj. self-dual} \\ \text{reg. alg.} \\ y \text{ place of } L\end{array}\right\} \Rightarrow \Pi_y$ is tempered (Ram.-Pet. conj. for GL_n)

- proved by Clozel at unram. pls...

<u>Idea:</u> Work with Sh. variety X whose coh. realizes $Re(\Pi)^{\otimes 2}$
- unit grp. looks like

$$U(1,n-1) \times U(1,n-1) \times U(0,n)^{d-2}$$

at ∞

quasi-split at all finite places

global inv is $\frac{nd}{2} + 2$ - even

X int. model for Sh. var., Y - special fiber

Ⓒ formula for $H^*(X, \mathcal{L}_\xi)$ in terms of cohom. of closed Newton polygon strata in Y

$$H^*(Y_{S,T}, \mathcal{L}_\xi)$$

param. for NP

Generalizes R.Z. space, Regular for s.s. scheme. Two spectral seq. converging to $H^*(X, \mathcal{L}_\xi)$.

Ⓓ $H^j(Y_{S,T}, \mathcal{L}_\xi)[\Pi^{\infty}] = 0$ outside the middle dim.
$\Rightarrow$ both sp. seq. are conc. on diag. both deg at E_2.

Ⓔ formula for cohom. of Ig. varieties
- covers open NP strata

$E \Rightarrow B$; $\left.\begin{matrix} B \\ E \end{matrix}\right\} \Rightarrow \left.\begin{matrix} D \\ C \end{matrix}\right\} \Rightarrow \left.\begin{matrix} A \\ B \end{matrix}\right\} \Rightarrow$ Thm

T.Y. style comp.

III Geometric result:

F' - CM field, $F' = F_1 E$ s.t. p split in E.
(F_1 tot. real; E im. quad)

$\wp$ prime above p.

$F = F_2 E$ $\quad F_2/F_1$ tot. real quad. ext.
$\wp = \wp_1 \wp_2$ splits in F
Work with $\wp_1, \wp_2$ simultaneously.
(corresp. embeddings $\tau_i : F \hookrightarrow \mathbb{C}$ $i = 1, 2$ where group G has sign $(1, n-1)$)

$\Pi' = \Psi \times BC_{F/F'}(\Pi)$. Can assume $\Pi_\wp$ has Iw. fixed vectors.
X = system of Sh. varieties$/F'$ ← reflex field.

ξ - irred. alg. rep. of $G/\overline{\mathbb{Q}}_\ell \rightsquigarrow \mathcal{L}_\xi$ $\quad \ell \neq p$.

$$\mathrm{Gal}(\overline{F}/F) \circlearrowright H^i(X, \mathcal{L}_\xi) = \bigoplus \pi \otimes R^i_{\xi,\ell}(\pi)$$
$$\circlearrowleft G(\mathbb{A}^\infty)$$

here we see $R_\ell(\Pi)^{\otimes 2}$

$\mathcal{U}$ = Iw level st. at $\wp_1, \wp_2$

$X_{\mathcal{U}}/\mathcal{O}_K$ - int. model w/ Iw level structure

$$K = F_{\wp_1} \simeq F_{\wp_2} \simeq F'_\wp.$$

$\mathcal{G}_i = A[\wp_i^\infty]$ $\quad i = 1, 2$

↑ one dim. compact Barsotti-Tate $\mathcal{O}_K$-module.

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$$\mathcal{C}_i : \mathcal{G}_i = \mathcal{G}_i^0 \to \mathcal{G}_i^1 \to \cdots \to \mathcal{G}_i^n \simeq \mathcal{G}_i / \mathcal{G}_i[\varpi].$$

$i = 1, 2$

Locally, $X_\mathcal{U}$ is étale over

$$X_{n,s} = \operatorname{Spec} \mathcal{O}_K[x_1, \dots, x_n, y_1, \dots, y_s] / (x_1 \cdots x_n - \varpi, y_1 \cdots y_s - \varpi)$$

prod. of s.s. schemes

(no longer s.s., but it is log smooth)

Globally, $i = 1, 2$ $\quad Y = \cup Y_j^i$ ← closed subsch. where j^{th} isogeny in $\mathcal{C}_i$, induces 0 map on Lie algebra (ex. Frob. conn. kernel)

$S, T \subseteq \{1, \dots, n\}$ nonempty

$$Y_{S,T} = \left(\bigcap_{i \in S} Y_i^1\right) \cap \left(\bigcap_{j \in T} Y_j^2\right) \quad \text{proper smth} / \overline{\mathbb{F}}$$

$$\dim 2n = \#S - \#T.$$

$$Y^{(l_1, l_2)} = \coprod_{\substack{\#S = l_1 \\ \#T = l_2}} Y_{S,T}.$$

Want to understand

$$N \curvearrowright H^*(X \times_K \overline{K}, \overline{\mathbb{Q}}_\ell) = H^*(Y \times_{\mathbb{F}} \overline{\mathbb{F}}, R\Psi \overline{\mathbb{Q}}_\ell)$$

← complex of nearby cycles.

$$N \curvearrowright R\Psi \overline{\mathbb{Q}}_\ell.$$

__Thm ©:__ There are G_K-equiv. spectral seq.

ker fil of N

$$E_1^{n, m-n} = \bigoplus_{p - q = n} H^m(Y_{\overline{\mathbb{F}}}\, Gr_I^q\, Gr_p^K\, R\Psi \overline{\mathbb{Q}}_\ell)$$

im fil. of N

$$\Rightarrow H^m(Y \times_{\mathbb{F}} \overline{\mathbb{F}}, R\Psi \overline{\mathbb{Q}}_\ell)$$

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and

$$(E_1')^{k+1,\, m-k+1} = \bigoplus_{i=1}^{p+q} H^{m-2k-p-q+1}\left(Y^{(k+i,\, k+p+q-i+1)}, \overline{\mathbb{Q}}_\ell(-(k+p-1))\right)$$

$$\Rightarrow H^m(Y, Gr^q_I\, Gr^k_\rho\, R\Psi\overline{\mathbb{Q}}_\ell)$$

Notes:
- E_1 = spectral seq. assoc. to mono. fil. of $R\Psi\overline{\mathbb{Q}}_\ell$.
- can rewrite them with $\mathcal{L}_\xi$ - coeff.
- generalizes R.Z. wt sp. seq.

Ingredients in proof:

① $X_U/\mathcal{O}_{K,M}$ log smooth over $(\mathcal{O}_K, N)$
↳ get explicit description of $R^k\Psi\overline{\mathbb{Q}}_\ell$ in terms of M w/ trivial I_K-action (C. Nakayama).

② Illusie's prod. form. for nearby cycles.

③ Saito's const. of wt. sp. seq. in semi stable case.

IV Computing Cohomology

D. $BC^p(H^j(Y_{S,T}, \mathcal{L}_\xi)[\pi^{1,\infty}]) = 0$

$$\circlearrowleft\ G(\mathbb{A}^{\infty,p})$$

$Y_{S,T} = \cup Y^0_{S,T'}$

open NP strata covered by towers of Igusa

(E) $\Rightarrow$ (B) (same (E) + (B) $\Rightarrow$ (D))

Assume $\Pi_\wp$ not tempered (e.vs of ϕ are $q^{\mathbb{Z}}$-numbers)

$\iota_p^{-1} \mathcal{L}_{n,\kappa}(\Pi_\wp)^{\otimes 2}$ shows up in $H^*(X, \mathcal{L}_\xi)$

Tadic, Jacquet-Shalika

- $\Pi_\wp$ tempered

or $\Pi_\wp = $ -nInd $(\pi_1 |\det|^{1/4} \times \pi_1 |\det|^{-1/4} \times \quad)$

$Sp_i(\pi_i)$ — Introduces an error factor

$$BC^p(H(\psi_{ST}, \mathcal{L}_\xi)[\pi]^{1,5})$$

$$= c \sum ([V^1_{j_1,j_2}] \oplus [V^2_{j_1,j_2}] \oplus [V^3_{j_1,j_2}])$$

$$V^k_{j_1,j_2} = \mathrm{rec}(\chi_{j_1}^{-1} \chi_{j_2}^{-1} | \cdot |^{?} \psi_u)$$

can't read this

$\varepsilon^k = -1/2, 0, 1/2.$

wts on RHS are $m-1, m, m+1$
$c \neq 0$
$\Big\}$ #

LHS alt sum

$\Rightarrow$ $\Pi_\wp$ is tempered.