Solvable Points on Curves:

(One variable)

$f(x) \in \mathbb{Q}[x]$

Theorem (Abel-Ruffini 1824): For $\deg(f) \geq 5$, the roots are not solvable.

(two variables)

Def $f: X_{/k} \to \mathbb{P}^1_{/k}$ is solvable if $K(t) \subset K(X)$ is solvable.

Conjecture (Enriques 1897 Congress): For genus$(X) > 6$, no such f exists

Theorem (Zariski 1926): (i) Proves the conjecture.
(ii) claims that for genus ≤ 6 such an f exists.

over $\mathbb{C}$

Mais (1964) Analytic proof of existence
Marten (1967) Set of f's with desired property is algebraic

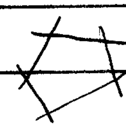
Theorem (Corella, et. at 1985):
Genus 5 : 5 f's , S_5 action
Genus 6 : family forms a curve (double cover of a degree 5 plane curve)

So no obvious way of extending this to the rationals.

Ambers Pal (2004)
(i) $g = 0, 2, 4$: $\exists$ solvable point
(ii) for $g \in \{6, 8, \ldots, \text{all } g \geq 40\}$, $\exists$ a curve over a local

field with no solvable points. Pick a local field s.t. it has a residue field with nonsolvable extensions, e.g. S_5. Then construct a stable curve s.t. the reduction has 5 components that are permuted by the Galois action, S_5 action.

over residue field



rational curve with S_5 action.

Motivation:

$X(N)$: modular curve of level N

$X(\rho)$: twisted curve classifying elliptic curves E s.t. $E[N] \simeq \rho$ where $\rho : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{Z}/N\mathbb{Z})$.

Choose $N = p_1 p_2$.
Choose ρ_{p_2} to be induced.
Suppose $X(\rho)$ has a point over a totally real solvable extension F. Then get E/F s.t. $E[N] \simeq \rho$.
ρ_{p_2} is modular (induced - hypothesis) $\Rightarrow$ E/F is modular.
$\Rightarrow$ ρ_{p_1} is modular.

If F is solvable, then any lift of ρ_{p_1} over $\mathbb{Q}$ is modular.

$g = 1$: (with Ciperiani)

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C over $\mathbb{Q}$,

Theorem: C genus $1/_{\mathbb{Q}}$. Then C has solvable points if

(i) $\mathrm{Pic}^0(C) = E$ has semi-stable reduction

(ii) $C(\mathbb{Q}_p) \neq \emptyset$ for all p.

$$0 \longrightarrow E(\mathbb{Q})/p^n E(\mathbb{Q}) \xrightarrow{\varphi} H^1_{Sel}(\bar{\mathbb{Q}}/\mathbb{Q}, E_{p^n}) \longrightarrow Ш_{p^n} \longrightarrow 0$$

$$H^1_{Sel}(\bar{\mathbb{Q}}/\mathbb{Q}, E_{p^n}) = \{ c \in H^1(\bar{\mathbb{Q}}/\mathbb{Q}, E_{p^n}) : c_\ell \in \mathrm{im}\,\varphi_\ell \ \forall \ell \}$$

$$(Ш)_{p^n} = Ш_{p^n}$$

Elements of Ш correspond to curves C as given in (i) & (ii).

$$\begin{array}{ccccccccc} 0 & \to & E(F)/p^n E(F) & \longrightarrow & H^1_{Sel}(\bar{F}/F, E_{p^n}) & \longrightarrow & Ш_{F,p^n} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & E(\mathbb{Q})/p^n E(\mathbb{Q}) & \longrightarrow & H^1_{Sel}(\bar{\mathbb{Q}}/\mathbb{Q}, E_{p^n}) & \longrightarrow & Ш_{\mathbb{Q},p^n} & \to & 0 \end{array}$$

Heegner construction:

$\exists \quad X_0(N) \longrightarrow E$

K imaginary quadratic field $\quad N = \mathfrak{n}\bar{\mathfrak{n}}$

$$\begin{array}{ccc} (E/\mathcal{O}_K, \ker \mathfrak{n}) & \longmapsto & \text{image in } E(H_K) \\ & & \big\downarrow \text{trace} \\ & & E(K) \end{array}$$

This is not practical in general. It is believed if rk $E > 1$ the trace always gives 0.

Theorem (Cornut-Vatsal): Get nonzero points over $E(K_n)$ this way (in anticyclotomic tower)

Construct classes in $H^1(\bar{K}_n/K_n, E_{p^{m_n}})$ for some $m_n \geq n$.

Want to construct

$$\varinjlim H^1_{Sel}(\bar{K}_n/K_n, E_{p^{m_n}})$$

using these Heegner classes.

$$0 \longrightarrow \varinjlim H^1_{Sel}(\bar{K}_n/K_n, E_{p^{m_n}})^{Gal(K_n/K)} \longrightarrow \text{``}\varinjlim\text{''} H^1_{\mathbb{Z}_n}(\bar{K}_n/K_n, E_{p^{m_n}})$$

$$\longrightarrow \varinjlim H^2(\bar{K}_n/K_n, E_{p^{m_n}}) \longrightarrow 0$$

$$\cong \hat{\Lambda}^{r_n}$$

$$\Lambda = \mathbb{Z}_p[[Gal(K_\infty/K)]]$$

Why do we get the unramified classes?

Method basically works in ordinary case.

In supersingular case:

Heegner point up the anticyclotomic tower

$$trace^{n-1}_n(x_n) = x_{n-2}$$

[Kobayashi & Perrin-Riou
Iovita & Pollack] Same property for local points } Known only if base field is unramified.