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Potential Automorphy for Compatible Systems of ℓ-adic Representations:

Joint work with Tom Barnet-Lamb, Toby Gee, and David Geraghty.

Weakly compatible system of ℓ-adic representations of $G_{\mathbb{Q}}$ consists of

- $\#S < \infty$ of bad primes
- M = number field of coefficients
- $\forall\ \lambda$ prime of M, $r_\lambda : G_{\mathbb{Q}} \longrightarrow GL_n(\bar{M}_\lambda)$ continuous and semi-simple.
- $\forall\ p \notin S$, $\lambda \nmid p \Rightarrow r_\lambda$ is unramified at p, $\mathrm{char}_{r_\lambda(\mathrm{Frob}_p)}(x) \in M[x]$ indep. of λ.
- r_λ is de Rham and crystalline if $\lambda \notin S$. Hodge-Tate numbers independent of λ.

Theorem: Suppose $\{r_\lambda\}$ is a weakly compatible system s.t.

1) (Strongly irreducible) Each r_λ is irreducible after restriction to any open subgroup. (ok to just assume for all but finitely many λ).
2) (regular) The Hodge-Tate numbers are all distinct.
3) (odd, ~~self~~-dual) ~~Each~~ r_λ either factors through GSp_n with odd multiplier $(c \mapsto -1)$ or factors through $GO(n)$ with even multiplier $(c \mapsto 1)$.

Then $\exists\ F/\mathbb{Q}$ a Galois totally real field such that $\{r_\lambda|_{G_F}\}$ is automorphic (π $GL_n(\mathbb{A}_F)$ regular algebraic essentially self-dual representation (RAESDC)).

There is no reason to restrict to $G_{\mathbb{Q}}$, could do G_F for F totally real. It is also possible for F a CM field, ~~though~~ it is not written down yet.

One gets from this that if one chooses an embedding $i : M \hookrightarrow \mathbb{C}$, then

$$L^S(i(\{r_\lambda\}), s)$$

has meromorphic continuation to $\mathbb{C}$, functional equation, etc.

"Pf": Choose a suitable λ and work with r_λ.
Find a "motive" X/F so that $H(X, \bar{\mathbb{F}}_\ell) \cong \bar{r}_\lambda$
and $H(X, \bar{\mathbb{F}}_{\ell'}) \cong \mathrm{Ind}_{G_M}^{G_F}(\bar{\theta}|_{G_M})$, M/F cyclic CM
deg n, $\theta : G_M \to \bar{\mathbb{Q}}_{\ell'}^{\times}$.
One knows from the theory of automorphic induction that
$\mathrm{Ind}(\theta)$ is automorphic. $\therefore$ $\mathrm{Ind}(\bar{\theta}) = H(X, \bar{\mathbb{F}}_{\ell'})$.
One uses Automorphy Lifting Theorems (ALT) of Wiles (analogous ones that is) to get that
$H(X, \bar{\mathbb{Q}}_\ell) \cong \bar{r}_\lambda$ is automorphic. Again apply the
automorphy lifting theorems to obtain that r_λ is
automorphic. $\square$

Some difficulties...

$X_t/T \qquad t = t_0$

large monodromy

consecutive H-T numbers + regular $\Rightarrow$ $0, 1, \ldots, n-1$.

Requirements on ALT: ① change the weight, i.e., H-T numbers.
② Can work over $F/\mathbb{Q}$ ramified at ℓ.

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Automorphy Lifting Theorems:

F totally real or CM

$r: G_F \to GL_n(\mathcal{O}_{\bar{\mathbb{Q}}_\ell})$, $\quad \bar{r}$ the reduction

Basic ALT:

① r unramified a.e. } Fontaine-Mazur says these should be

② r deRham } enough

③ r is regular

④ r odd, self-dual

⑤ $\bar{r}(G_{F(\zeta_\ell)})$ is "big", need $\ell > n$ and $\zeta_\ell \notin \bar{F}^{\ker \operatorname{ad} \bar{r}}$.

⑥ $\exists\, r_0$ with $\bar{r}_0 = \bar{r}$ with r_0 automorphic and r_0 and r define points on same irred. component of $\operatorname{Spec}(R^{\square}_{\bar{r}|_{G_{F_v}}} \otimes \bar{\mathbb{Q}}_\ell)$ $\forall\, v \nmid \ell$ (This is only a condition on finitely many primes) and

> Ex: $n = 2$,
>
> $$\bar{r}|_{G_{F_v}} = \begin{pmatrix} \bar{\epsilon}_\ell & 0 \\ 0 & 1 \end{pmatrix}, \quad \ell \nmid p^2 - 1$$
>
> two ~~component~~ $\rightarrow$ nr component / $\rightarrow$ mult. non-trivial unip. monodromy

$$\operatorname{Spec}(R^{\square}_{\bar{r}|_{G_{F_v}}, \underbrace{cr}_{\text{crys.}}, \underbrace{H}_{\text{fixed H-T \#'s}}} \otimes \mathbb{Q}_\ell) \qquad \forall\, v \mid \ell.$$

Then r is automorphic.

- $F_v \mid \mathbb{Q}_\ell^{\text{unr}}$, $H \subset [0, \ell-2]$ 1 irred. comp. [CHT]
- $\exists \le 1$ ordinary, crystalline component (Geraghty)

- $H = \{0, 1\}$, $n = 2$

  2 - comps. $\begin{cases} \text{ord} \\ \text{s.s} \end{cases}$ (Kisin)

Def: $v \mid \ell$, $r : G_{F_v} \longrightarrow GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_\ell})$ is potentially diagonalizable (potentially crystalline)

if $\exists\, K/F_v$ finite ext. s.t. $r|_{G_K}$ is on the same component of the local deformation ring as $\chi_1 \oplus \cdots \oplus \chi_n$ chars.

Examples: ① ordinary

② crystalline in FL - range

③ r pot. diag. $\Rightarrow r|_{G_{F_w'}}$ is pot. diag.

Thm: r unram. a.e., de Rham, regular, odd self-dual,

⑤ $\bar{r}(G_{F(\zeta_\ell)})$ is $2n$ - big, $[\bar{F}^{\ker \mathrm{ad}\, \bar{r}}(\zeta_\ell) : \bar{F}^{\ker \mathrm{ad}\, \bar{r}}] > n$,

⑥ $r|_{G_{F_v}}$ is cryst. and pot. diag., $\bar{r}$ has an auto. lift r_0 which is also cryst. and pot. diag.

Then r is auto.

"Pf": r $\quad r_0 \quad r_0$ auto $\quad \bar{r} = \bar{r}_0$.

$r|_{G_{F_v}} \sim \chi_1 \oplus \cdots \oplus \chi_n$ means on same component

$r_0|_{G_{F_v}} \sim \chi_{0,1} \oplus \cdots \oplus \chi_{0,n}$.

$M \mid F$ cyclic CM deg. n.

θ, θ_0 chars. of G_M $\quad \bar{\theta} = \bar{\theta}_0$

$\mathrm{Ind}_{G_M}^{G_F} \theta|_{G_{F_v}} \sim \chi_1 \oplus \cdots \oplus \chi_n$

$\mathrm{Ind}_{G_M}^{G_F} \theta_0|_{G_{F_v}} \sim \chi_{0,1} \oplus \cdots \oplus \chi_{0,n}$.

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$r \otimes \mathrm{Ind}\,\Theta_0$ $\qquad$ $r_0 \otimes \mathrm{Ind}\,\Theta$
auto

same red.

basic ALT $\Rightarrow r \otimes \mathrm{Ind}\,\Theta_0$ ~~auto~~ $\Rightarrow r$ auto.
$\square$.

The tensoring with Ind. originated with Michael Harris, though to solve a different problem.