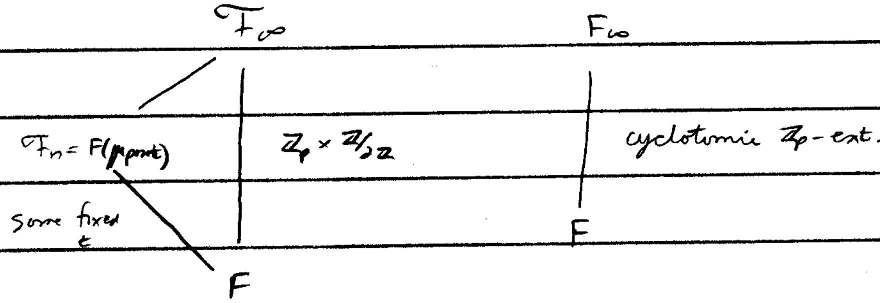
Khare

3-22-10

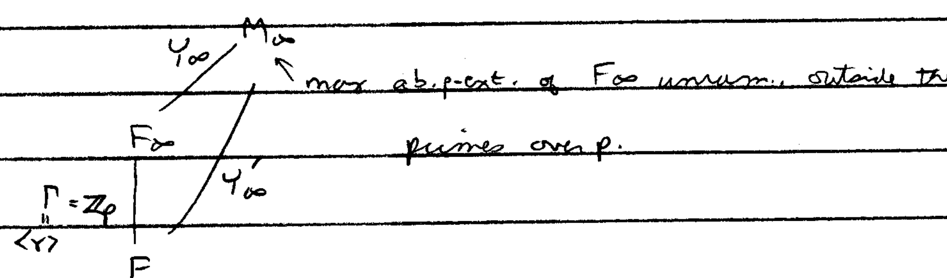
Extension of Iwasawa Modules (I) :

Joint work with J.P. Wintenberger.

- F totally real field, $p>2$ prime
- $[F(\mu_p):F]=2$
- $\mathcal{F}_\infty = F(\mu_{p^\infty})$ $\quad Gal(\mathcal{F}_\infty/F) \cong \mathbb{Z}_p \times \mathbb{Z}/2\mathbb{Z}$



Leopoldt's conjecture: F_∞/F is the unique $\mathbb{Z}_p$-ext of F.



$$0 \to Y_\infty \to Y_\infty' \xrightarrow{\text{degree}} \mathbb{Z}_p \to 0$$

{ max. ab. qt.

$$0 \to Y_\infty/(\gamma-1)Y_\infty \to Y_\infty'/(\gamma-1)Y_\infty \to \mathbb{Z}_p \to 0$$

$$\| $$

$$Gal(F^{ab,p}/F)$$

where $F^{ab,p}$ is the max. ab. p-ext. of F unram. outside p.

Leopoldt's conj. $\Leftrightarrow$ $Y'_\infty/(\gamma-1)Y_\infty$ has $\mathbb{Z}_p$ rank one

$\Leftrightarrow$ $Y_\infty/(\gamma-1)Y_\infty$ is a finite group

$\Leftrightarrow$ $\underbrace{H^1(\Gamma, Y_\infty)}$ is a finite group.

want to write down
some elements of this group.

Reciprocity conjecture: X/k smooth proj. curve.

$P, Q \in X(k)$

$X_{P,Q}/k$ α

$$0 \to \mathbb{G}_m \to \mathrm{Jac}(X_{P,Q}) \to \underset{\overset{\|}{J}}{\mathrm{Jac}(X)} \to 0$$

$$\alpha_{P,Q} \underset{(P)-(Q)}{\overset{\|}{}} \in \mathrm{Ext}^1(J/\mathbb{G}_m) = J$$

We ask for an Iwasawa theoretic analogue.

Iwasawa pairing: $F_n = F(\mu_{p^{n+1}})$

$A_n^- = \mathrm{Syl}_p(\mathrm{Cl}_{F_n})^-$

$\varinjlim_n A_n^- = A_\infty^-$.

Perfect $\underset{\substack{\shortparallel \\ Gal({}^{\sigma}F_\infty/F)}}{\Gamma} \times \mathbb{Z}/2\mathbb{Z}$ equivariant pairing:

$$\begin{array}{c} Y_\infty \times A_\infty^- \longrightarrow \mu_{p^\infty} \\ \text{\tiny$\cup$} \\ (\sigma, \ \operatorname{im} c_n) \end{array} \qquad\qquad {}^{\sigma}F_\infty(\sqrt[p^m]{\alpha}) \atop {\Big| \atop {}^{\sigma}F_\infty}$$

$c_n \in A_n^-$, $c^{p^m} = (\alpha)$

$$\longrightarrow \frac{\sigma(\sqrt[p^m]{\alpha})}{\sqrt[p^m]{\alpha}}$$

$$Y_\infty = \operatorname{Hom}_{\mathbb{Z}_p}(A_\infty^-, \mu_{p^\infty})$$

$$\therefore \quad H^1(\Gamma, Y_\infty) = H^1(\Gamma, \operatorname{Hom}(A_\infty^-, \mu_{p^\infty})).$$

Consider a finite set of primes $Q = \{q_1, q_2, \ldots, q_m\}$ of F s.t. q_i is inert in F_∞/F. $m = \#Q$.

Q_n = set of primes of ${}^{\sigma}F_n$ above Q.

By abuse of notation, we also denote by Q_n the product of the primes in Q_n.

Consider A_{n,Q_n}^- = $-$part of Sylow p-subgroup of the ray class group of ${}^{\sigma}F_n$ of conductor Q_n.

One has an exact sequence

$$0 \to \mu_{p^{n+t}}^{m-1} \longrightarrow A_{n,Q_n}^- \longrightarrow A_n^- \to 0.$$

Now take direct limits and obtain

$$0 \to \mu_{p^\infty}^{m-1} \to A^-_{\infty,Q} \to A^-_\infty \to 0$$

where $A^-_{\infty,Q} = \varinjlim A^-_{n,Q_n}$.

Restrict to $m=2$: $Q = \{q_1, q_2\}$

$$0 \to \mu_{p^\infty} \to A^-_{\infty,Q} \underset{\underset{f \text{ section}}{\curvearrowleft}}{\longrightarrow} A^-_\infty \to 0.$$

Consider $\mathrm{Hom}(A^-_\infty, ?)$ (using that μ_{p^∞} is divisible the seq. remains exact, then take long exact seq.)

$$0 \to \mathrm{Hom}(A^-_\infty, \mu_{p^\infty})^\Gamma \to \mathrm{Hom}(A^-_\infty, A^-_{\infty,Q})^\Gamma$$
$$\to \mathrm{Hom}(A^-_\infty, A^-_\infty)^\Gamma \xrightarrow{\delta} H^1(\Gamma, \mathrm{Hom}(A^-_\infty, \mu_{p^\infty}))$$

$\delta(id) = c_Q$ = class of $\gamma \mapsto \gamma\cdot f - f. \in \mathrm{Hom}(A^-_\infty, \mu_{p^\infty})$.

$\mathbb{Z}_p \cdot c_Q \leftarrow$ line

Want a "formula" for the line $\mathbb{Z}_p \cdot c_Q$.

Motivation (geometric):

$$0 \to \mathbb{G}_m \to \mathrm{Jac}(X_{P,Q}) \to \mathrm{Jac}(X) \to 0.$$

$\Downarrow$ l-adic Tate modules, $l \neq \mathrm{char}(k)$

$$0 \to \mathbb{Z}_l(1) \to Ta_l(\mathrm{Jac}(X_{P,Q})) \to Ta_l(\mathrm{Jac}(X)) \to 0.$$

$$(\alpha_{P,Q})_l \in H^1(G_k, \mathrm{Hom}(Ta_l(\mathrm{Jac}(X)), \mathbb{Z}_l(1))$$
$$\overset{\text{Weil pairing}}{\|}$$
$$H^1(G_k, Ta_l(\mathrm{Jac}(X))).$$

Khare
3-22-10

$$\varprojlim_n J(k)/\ell^n J(k) \xrightarrow{\delta} H^1(G_k, T_\ell(Jac(X))).$$

$$(\alpha_{P,Q})_\ell = \delta((P)-(Q))$$

Back to our case:

$$H^1(\Gamma, Y_\infty) = H^1(\Gamma, \mathrm{Hom}(A'_\infty, \mu_{p^\infty}))$$

$$\underset{\mathbb{Z}_p \cdot c_\infty}{\Psi}$$

$$\| \quad Y_\infty/(\gamma-1)Y_\infty$$

Points in geometric case $\longleftrightarrow$ Frobenius elements in arithmetic case.

$$Y'_\infty/(\gamma-1)Y_\infty \cong Gal(F^{ab,p}/F)$$

$Q = \{q_1, q_2\}$.

$$\rightsquigarrow$$

$$\mathrm{Frob}_{q_1}, \mathrm{Frob}_{q_2} \in \underbrace{Gal(F^{ab,p}/F)}_{\mathbb{Z}_p\text{- module}}$$

Consider the $\mathbb{Z}_p$- submodule M_Q of $Y'_\infty/(\gamma-1)Y_\infty$ generated by Frob_{q_i} $i=1,2$.

Define N_Q to be the degree 0 submodule of M_Q, so we have:

$$0 \to N_Q \to M_Q \to \mathbb{Z}_p$$

Note that N_Q is cyclic.

Reciprocity conjecture: $N_Q = \mathbb{Z}_p \cdot c_Q$

Compatibility of reciprocity conjecture with Leopoldt conjecture:

$$\varprojlim_n A_n^- = X_\infty^-$$

$$\varprojlim_n A_{n,Q_n}^- = X_{\infty,Q}^-$$

By class field theory,

$$\begin{tikzcd}
 & & L_\infty^-(Q) \\
 & L_{\infty,-}^- \arrow[ur, dash, "\mathbb{Z}_p(1)"] & \\
F_\infty \arrow[ur, dash, "X_\infty^-"] \arrow[uurr, dash, "X_{\infty,Q}^-"'] & & \\
F \arrow[u, dash, "\mathbb{Z}_p \times \mathbb{Z}/2\mathbb{Z}"'] & &
\end{tikzcd}$$

$L_{\infty,-}^-$ = max. ab. p-ext. of F_∞ unram everywhere

Get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_p(1) \arrow[r] & X_{\infty,Q}^- \arrow[r] & X_\infty^- \arrow[r] \arrow[l, bend left, dashed, "f"] & 0
\end{tikzcd} \qquad (*)$$

Iwasawa theory of adjoints:

$$(X_\infty^-)^\circ \sim \mathrm{Hom}(A_\infty^-, \mathbb{Q}_p/\mathbb{Z}_p).$$

pseudo-isom.

invert action of Γ

Leopoldt's conjecture $\Rightarrow$ $(*) \otimes \mathbb{Q}_p$ splits (i.e. $(*)$ splits up to isogeny.) (as Γ-modules)

The splitting is equivalent to the existence of a Kummer extension

$$\begin{tikzcd}[column sep=small]
N_\infty \arrow[dd, dash] \arrow[dr, dash, "\mathbb{Z}_p(1)"] & \\
& \mathcal{F}_\infty \arrow[dl, dash, "\mathbb{Z}_p \times \mathbb{Z}/2\mathbb{Z}"] \\
F &
\end{tikzcd}$$

s.t. $N_\infty / \mathcal{F}_\infty$ is ramified only at primes of $\mathcal{F}_\infty$ above

Q.

$$\begin{tikzcd}[column sep=small]
N_\infty \arrow[dr, dash, "\mathbb{Z}_p(1)"] & & \mathcal{L}_\infty^- \arrow[dl, dash, "X_\infty^-"'] \\
& \mathcal{F}_\infty \arrow[d, dash, "\mathbb{Z}_p \times \mathbb{Z}/2\mathbb{Z}"] & \\
& F &
\end{tikzcd}$$

Prop: $[N_\infty \cap \mathcal{L}_\infty^- : \mathcal{F}_\infty]$ divides $|N_Q|$.
(assuming Leopoldt's conjecture.)
under other mild hypothesis $\mu_p(F_v) = 1$ $\forall$ $v | p$,
in fact there is an equality.

"Pf": CFT:

$$0 \to \underline{U^1_{F,p}} = \prod_{v|p} U^1_{F_v}$$

Khare
3-22-10
p98

$$0 \to U^1_{F,p}/\overline{E}'_F \to \mathrm{Gal}(F^{ab,p}/F) \to \mathrm{Syl}_p(Cl_F) \to 0$$

The proof comes about by thinking of $U^1_{F,p}/\overline{E}'_F$ "Kummer theoretically". □