Emerton

3-25-10

pg 1

On the Iwasawa Theory of Modular Forms:

Joint work w/ R. Pollack and T. Weston

Some formalism of Θ-elements:

E finite over $\mathbb{Q}_p$ (field of coefficients)

$\cup\!\shortmid$

$\mathcal{O}$ ring of integers

$\Pi/\mathcal{O}$ is a rep. of $GL_2(\mathbb{Q}_p)$.

$\Gamma = \begin{pmatrix} \mathbb{Z}_p^\times & \\ & 1 \end{pmatrix}$

$\shortparallel$

Γ_0

$n \geq 1, \quad \Gamma_n = \begin{pmatrix} 1+p^n\mathbb{Z}_p & \\ & 1 \end{pmatrix} \subseteq \Gamma$

$P_0 = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$

Let $v \in \Pi^{P_0}$ (vector fixed by P_0).

Define $c_n(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} v \in \Pi^{\Gamma_n}$

$\Pi^* := \mathrm{Hom}_{\mathcal{O}}(\Pi, \mathcal{O})$.

$c_n(v) : (\Pi^*)_{\Gamma_n} \longrightarrow \mathcal{O}$

Γ_n - coinvariants.

We have an action of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ on $(\Pi^*)_{\Gamma_n}$, and so it is a $\Lambda_n := \mathcal{O}[(\mathbb{Z}/p^n\mathbb{Z})^\times]$ - module.

Λ_n - linear

$\Theta_n(v) : (\Pi^*)_{\Gamma_n} \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\Lambda_n, \mathcal{O}) \cong \Lambda_n$

$\ell \longmapsto (\lambda \longmapsto \langle \lambda\ell, c_n(v) \rangle)$

Λ_n - linear

$\Theta_n(v) : (\Pi^*)_{\Gamma_n} \longrightarrow \Lambda_n \qquad \Lambda_n \times \Lambda_n \xrightarrow{m} \Lambda_n \xrightarrow{\text{coeff of } 1} \mathcal{O}$

$\Theta_n(v) : \ell \longmapsto \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \langle \ell, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} v \rangle$

$\cap$

$(\Pi^*)_{\Gamma_n}$

with $(\mathbb{Z}/p^n\mathbb{Z})^\times \longleftrightarrow \Lambda_n^\times$

$a \longmapsto \sigma_a.$

$$\begin{tikzcd}
(\Pi^*)_{\Gamma_n} \arrow[d] \arrow[dr] & \\
(\Pi^*)_{\Gamma_{n-1}} \arrow[r, "C_{n-1}(v)"] & \mathcal{O}
\end{tikzcd}$$

denote this by $\mathrm{cores}^n_{n-1} C_{n-1}(v)$.

$$\begin{tikzcd}
(\Pi^*)_{\Gamma_n} \arrow[r, "C_n(v)"] & \mathcal{O} \\
(\Pi^*)_{\Gamma_{n-1}} \arrow[u, "\mathrm{cores}"] \arrow[ur] &
\end{tikzcd}$$

denote this by $\mathrm{tr}^n_{n-1} C_n(v)$.

One has that

Hecke operator $\sum_{i=0}^{p-1} \begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$.

$$\mathrm{tr}^n_{n-1} C_n(v) = C_{n-1}(h_{(p),N_0}\, v)$$

Now suppose we have $\rho : G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathcal{O})$ cont. s.t. $E \otimes_{\mathcal{O}} \rho_f$ is crystalline with H-T weights $(0, 1-k)$ $k \geq 2$. Can also just assume it is irreducible and won't lose much.

Suppose that $\bar{\rho}$ has no subquotient of the form $\omega^i, i \in \mathbb{Z}$ where ω is the mod p cyclotomic character. (we say $\bar{\rho}$ is not anomolous.)

We also assume that $\bar{\rho}$ is distinguished, i.e., is not and extension of a character by the same character $\begin{pmatrix}\chi & * \\ 0 & \chi\end{pmatrix}$.)

p-adic Local Langlands (Berger-Breuil) gives

$\Pi(\rho)$ p-torsion free, p-adically complete $\mathcal{O}$-module

$\circlearrowleft$

$GL_2(\mathbb{Q}_p)$

As Colmez's functor: $\Pi(\rho) \mapsto \rho$.

central char
$= \det \rho \cdot \varepsilon$
p-adic cyclo char.

$$(\mathrm{Sym}^{k-2}\mathcal{O})^\vee \hookrightarrow \Pi(\rho)$$

(k can be as big as you like here)

$GL_2(\mathbb{Z}_p)$-equiv. (saturated)

Ψ highest weight vector for $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ fixed by P_0 $\longmapsto v_{new} \in \Pi(\rho)^{P_0}$.

$\forall\, n \geq 0$, $c_n(v_{new}) : (\Pi(\rho)^*)_{\Gamma_n} \longrightarrow \mathcal{O}$

$\Theta_n(v_{new}) : (\Pi(\rho)^*)_{\Gamma_n} \longrightarrow \Lambda_n$.

<u>Modular forms:</u>

f is a normalized cuspidal newform of weight $k \geq 2$, conductor N s.t. $p \nmid N$, defined over E (more interesting to assume non-ord).

$$\rho_f : G_{\mathbb{Q}} \longrightarrow GL_2(E).$$

with

char poly of geo Frob at ℓ $= X^2 - a_f(\ell) X + \varepsilon(\ell)\ell^{k-1}$ $\quad \ell \nmid N$

Assume $\bar{\rho}_f^{ss}$ is irred. so ρ_f has a unique integral model

$$\rho_f : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O}).$$

Further assume

$$\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$$

is distinguished and not anomolous.

$$\rightsquigarrow \Pi(\rho_f) := \Pi(\rho_f|_{G_{\mathbb{Q}_p}}).$$

For any $r \geq 0$, $Y_1(N;p^r)$ = modular curve classifying $\mu_N \hookrightarrow E[N]$ full p^r-level structure, ϖ unif. of $\mathcal{O}$.

$$\tilde{H}^1_{c,\mathcal{O}} := \varprojlim_s \varinjlim_r H^1_c(Y_1(N;p^r), \mathcal{O}/\varpi^s) \quad (\text{étale cohom} / \bar{\mathbb{Q}})$$

$$\tilde{H}^1_{c,E} = E \otimes_{\mathcal{O}} \tilde{H}^1_{c,\mathcal{O}}.$$

$$\mathbb{T}[G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)]$$

$$\tilde{H}^1_{c,\mathcal{O}} \xrightarrow{\{0,\infty\}} \mathcal{O}$$

$$E \qquad\qquad E$$

$$f \rightsquigarrow \lambda_f : \mathbb{T} \longrightarrow \mathcal{O} \qquad (\mathcal{O}\text{-valued pt. of Spec } \mathbb{T}).$$

$$\rho_f \otimes \Pi(\rho_f) \leq H^1_{c,\mathcal{O}}[\lambda_f] = \lambda_f\text{-eigenspace}$$

(Breuil, Buzzard + E.)

(actually is equality, but harder to prove!)

Assume $p>2$. Fix a sign $\pm$. So we get

eigenvalue of complex. conj.

$$\begin{tikzcd} \Pi(\rho_f) \arrow[r] \arrow[rr, bend right, "\ell_f"'] & \tilde{H}^{1,\pm}_{c,0} \arrow[r, "\{0,\infty\}"] & \mathcal{O} \end{tikzcd}$$

$$\Theta_n(v_{new}) : (\Pi(\rho_f)^*)_{\Gamma_n} \longrightarrow \Lambda_n .$$

<u>Prop:</u> If $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \mathcal{O}^\times$ has conductor p^n $(n>0)$ and $\chi(-1) = \pm$ (to agree with choice above), then

$$\chi(\langle \ell_f, \Theta_n(v_{new})\rangle) = \frac{\tau(\chi)\, L(f,\bar{\chi},1)}{\Omega^{\pm}_{f,can}}$$

<u>Pf:</u> Birch lemma, (Kirillov model). ■

One has due to Colmez

$$\begin{array}{c} D(\rho_F)^{\psi=1} \cong (\Pi(\rho_f)^*)^{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} \\ \wr\| \text{ Fontaine} \\ H^1(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p, \rho_f^\vee) \ni \ell_f \\ \| \\ H^1_{Iw,loc}(\rho_f^\vee). \end{array}$$

$$\begin{tikzcd} H^1_{Iw,gl}(\rho_f^\vee) \arrow[r, "res."] & H^1_{Iw,loc}(\rho_f^\vee) \\ z_{Kato} \arrow[u, phantom, "\cup"] \arrow[r, squiggly] & z_{Kato,loc}. \end{tikzcd}$$

Kato's Euler system

still checking these ones

Prop:* If $\rho_f|_{G_{\mathbb{Q}_p}}$ is irred. then $Z_{\mathrm{Kato,loc}} = \ell_f$.

$$\begin{array}{ccc} C_n(v_{new}) : \left(H^1_{Iw,loc}(\rho_f^\vee)\right)_{\Gamma_n} & \longrightarrow & 0 \\ \big\downarrow \cong & & \\ H^1(\mathbb{Q}_p(\zeta_{p^n}), \rho_f^\vee) & & \end{array}$$

b/c $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is non-anomolous.

Lemma*: $C_n(v_{new})$ annihilates $H^1_f(\mathbb{Q}_p(\zeta_{p^n}), \rho_f^\vee)$

so

$C_n(v_{new}) \in H^1_f(\mathbb{Q}_p(\zeta_{p^n}), \rho_f(1))$. (Using duality)

(analogs in $k > 2$ of ~~Kobayashi~~ classes)

Theorem*: The ideal generated by

$(\mathrm{cor}^n_i \langle \ell_f, \Theta_i(v_{new}) \rangle)_{0 \le i \le n}$

ideal in Λ_n

$\subseteq \mathrm{Fitt}(\mathrm{Sel}(\mathbb{Q}_p(\zeta_{p^n}), E/\mathcal{O} \otimes \rho_f^\vee)^\vee)$.

(confiaming a conjecture of Mazur and Tate in this case..)

Pf: Kato's divisibility result & the classes $C_n(v_{new})$ & global duality & chasing Fitting ideals. ∎