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Special Values modulo p:

Non-vanishing of twists:

$\zeta(s)$ = Riemann zeta function

$\zeta(k) \in \mathbb{Q}$ if k neg odd.

$\zeta(-1) = \frac{-1}{12}$, $\zeta(-3) = \frac{1}{60}$

$\zeta(1-2k) = {-B_k}/{k}$ where $\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$

$\boxed{p \text{ odd prime}}$

Kummer: $h(\mathbb{Q}(\zeta_p))$ is divisible by $p \iff p \mid \mathrm{num}(B_k)$,

$k = 2, 4, 6, \ldots, p-3$.

$N > 2$ integer, $L(s, \chi)$ is a Dirichlet L-function, χ prim. mod N.

$$L(1-n, \chi) = \frac{-B_{n,\chi}}{n} \quad \text{where} \quad \frac{\sum_{a=1}^{N} \chi(a)\, t e^{at}}{e^{Nt} - 1} = \sum B_{n,\chi} \frac{t^n}{n!}.$$

$$h(\mathbb{Q}(\sqrt{-p})) = \frac{\sqrt{p}}{\pi} L(1, \chi) \sim B_{0,\chi}.$$

$\chi \in S$ = some family

How often is $B_{n,\chi}$ divisible by some fixed prime of $\overline{\mathbb{Q}}$ (res. char ℓ)

as χ varies?

Example: S = quad. chars. of cond (D).

p-adic families:

S = Dirichlet chars. of cond. p^n as $n \to \infty$.

Iwasawa: If e_n is the exponent of p in class number of $\mathbb{Q}(\zeta_{p^n})$,

then $e_n = \lambda n + \mu p^n + \nu$ for all $n \gg$, $\lambda, \mu, \nu \in \mathbb{Z}$.

Iwasawa conj: $\mu = 0$.

Exponent of $\ell \neq p$ in $h(\mathbb{Q}(\zeta_{p^n}))$?

The evidence was that $\mathrm{ord}_\ell(h(\mathbb{Q}(\zeta_{p^n}))) < c_{\ell,p}$.

Proved by Ferrero-Washington (1984).

Starting point: Explicit formulas for class numbers in terms of

Bernoulli numbers.

Formula (Iwasawa):

$B_n \sim$ related to digits (p-adic) of $p-1$ roots of unity.

Point: prove that digits of $p-1$ roots of unity behave like indep. random variables. ← first hint of probability ...

Recall: $a_0 + a_1 p + \cdots = \alpha \in \mathbb{Z}_p$ is called normal if every

string of length k of digits appears with frequency p^{-k}.

Easy to show set of non-normal elements has measure 0, but very hard

to determine if a specific one is normal or not.

Ferrero-Washington: Suppose $\gamma_1, \dots, \gamma_r$ are linearly indep. over $\mathbb{Q}$,

then for almost all $\beta \in \mathbb{Z}_p$ the sequence of vectors $\in [0,1]^r$

$$X_n(\beta) = (x_n(\beta\gamma_1), \dots, x_n(\beta\gamma_r)) \quad \text{is unif. distributed}$$

in $[0,1]^r$,

$$X_n(\alpha) = \frac{\text{unique integer in } [0, p^n) \equiv \alpha \pmod{p^n}}{p^n} .$$

Analogy: (Kronecker) Suppose $\gamma_1, \dots, \gamma_r \in \mathbb{R}$ are linearly indep. over $\mathbb{Q}$,

then $(t\gamma_1, t\gamma_2, \dots, t\gamma_r)$ has dense image (unif) in $[0,1]^r$.

This was reproved by Sinnott. He used Euler's formula that relates B_k to derivatives of rational functions.

$\mathbb{F}((T-1))$ = Laurent series in $T-1$. (over $\mathbb{F} = \overline{\mathbb{F}}_p$)

Sinnott: Suppose $\gamma_1, \dots, \gamma_r \in \mathbb{Z}_p$ are linearly indep. over $\mathbb{Q}$, then

$T^{\gamma_1}, \dots, T^{\gamma_r}$ are algebraically indep. in $\mathbb{F}((T-1))$,

$T^\alpha = \sum \binom{\alpha}{n} (T-1)^n$.

This seems at first glance to be an entirely algebraic proof.

$\mathbb{F}[[T-1]]$ power series in 1 variable. formal completion of $\mathbb{G}_m$.

$\mathbb{F}(T^{\gamma_1}, T^{\gamma_2}, \dots, T^{\gamma_r}) \subset \mathbb{F}((T-1))$

$\quad \cong$

poly ring in

r variables.

Geometrically: 1 parameter formal $\mathbb{G}_m$ dense inside

r-dim. space.

So it is actually similar to before. It seems all examples, at least ones looked at today, are something small mapping into something large and the image turns out to be dense.

Example: Hecke L-functions (Hida)

Key ingredient: Chai

k = alg. closed field of char $p > 0$, X = smooth finite dim. formal p-div. group over k.

$E_{\mathbb{Z}_p} = \mathrm{End}(X)$, $E = E_{\mathbb{Z}_p} \otimes \mathbb{Q}_p$ = finite dim. v.s. over $\mathbb{Q}_p$.

$\mathbb{E}$ = lin. alg. grp. over $\mathbb{Q}_p$ s.t. $\mathbb{E}(R) = (E \otimes_{\mathbb{Q}_p} R)^\times$, R any $\mathbb{Q}_p$-alg.

G any alg. group over $\mathbb{Q}_p$ and $\rho : G \to \mathbb{E}$ is a homom., can regard ρ as a rep. of G on E via $\mathbb{E} < \mathrm{Aut}(E)$.

Chai: Suppose that the trivial rep. is not a subquotient of ρ (of G on E). Suppose Z is a reduced irred. closed formal subscheme of X closed under action of an open subgroup of $G(\mathbb{Z}_p)$. Then Z is closed under group law of X and is a p-div. subgroup.

Anticyclotomic twists of GL_2 L-functions:

F = totally real field, K/F = imag. quad ext.

π Hilbert mod. form of wt $(2, \ldots, 2)$ for F.

Look at $L(\pi, \chi, s)$, χ anticycl. character of $\mathbb{A}_K^\times$

Means: χ factors through $\mathbb{A}_F^\times \subset \mathbb{A}_K^\times$.

central char. of π is unram. and $\chi\omega = 1$ on $\mathbb{A}_F^\times$.

Look at $L(\pi, \chi, \frac{1}{2})$ for χ running over anticycl. chars

of cond. $\mathfrak{p}^n$ as $n \to \infty$ where $\mathfrak{p}$ = prime of K.

Expect $L(\pi, \chi, \frac{1}{2}) \neq 0$ or to vanish to order 1 (typically).

Proved under mild assumptions (Cornut - V.). Don't want to spend time on the statement, rather on the ingredients that go into the proof and the analogies with stuff already discussed.

<u>Key ingredient:</u> Thm of M. Ratner

$G = SL_2(\mathbb{Q}_p)$, $\Gamma_i \subset G$ discrete, cocompact subgroups. say that Γ_1 and Γ_2 are <u>commensurable</u> if $\Gamma_1 \cap \Gamma_2$ has finite index in both.

<u>Thm (Ratner):</u> Suppose that Γ_1 and Γ_2 are <u>NOT</u> commensurable,

then $\Gamma_1 \cdot \Gamma_2 = \{\gamma_1\gamma_2 : \gamma_i \in \Gamma_i\}$ is dense in G.

not a group!

<u>Kronecker:</u> $G = \mathbb{R}$, Γ_1, Γ_2 discrete subgroups $\mathbb{Z}\gamma_1, \mathbb{Z}\gamma_2$, $\Gamma_1 \cdot \Gamma_2$ is dense iff γ_1, γ_2 are indep over $\mathbb{Q}$, $\Gamma_1 \cap \Gamma_2$ is trivial (not commensurable).

Note that the converse of Ratner's theorem is clear.

More generally, suppose $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ are pairwise not comm., then the image of the diagonal inside

$$X = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_r \backslash G \times \cdots \times G$$

is dense in X.

More generally still

G = any p-adic Lie grp

Γ = discrete subgroup s.t. $\Gamma \backslash G$ has finite measure. wrt the unique G-invariant Borel measure.

$H \subset G$ any subgroup generated by image of maps

$$U : (\mathbb{Q}_p) \longrightarrow G$$

then the closure $\overline{H}$ of H in $\Gamma \backslash G$ is homog $\exists\ H' \supset H$ s.t. $\mathrm{im}(H') = \overline{H}$.