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p1

p odd prime $\qquad T$ a $\mathbb{Z}[\frac{1}{N}]$-scheme

N integer prime to p.

$X = X_1(N)/_{\mathbb{Z}[\frac{1}{N}]}$ classifies elliptic curves $(E,i)/_T$

$$i : \mu_N/_T \to E/_T$$

$$\begin{array}{l} \mathbb{E} \\ \downarrow \Big) e \\ X \end{array} \qquad \omega = e^* \Omega_{\mathbb{E}/X}$$

$$k \geq 0 \qquad \omega^k/_X.$$

A $\quad \mathbb{Z}[\frac{1}{N}]$-algebra.

$$S_k(N,A) = H^0(X, \omega^k/_A(-\text{cusp}))$$

$$\cap \text{I}$$

$$f \in M_k(N,A) = H^0(X, \omega^k/_A).$$

$$(E,i,\omega)/A \;\longmapsto\; f(E,i,\omega) \in A.$$

q-expansion $\quad \mathrm{Tate}(q) = \mathbb{G}_m/q^{\mathbb{Z}} \,/ \mathbb{Z}_p(\!(q)\!)$

$f \in M_k(N,A) \qquad \omega_{can} \in H^0(\mathrm{Tate}(q), \Omega_{\mathrm{Tate}(q)/\mathbb{Z}_p(\!(q)\!)}).$

$$f(q) = f(\mathrm{Tate}(q), \omega_{can})$$

Hecke operators on $M_k(N,A)$.

$T_\ell, \; \langle \ell \rangle \qquad \ell \nmid N$

$U_\ell, \; \ell \mid N.$

- ordinary forms

$$T_p \circlearrowleft M_k(N,\mathbb{Z}_p). \quad , \quad e_0 = \lim_{n\to\infty} T_p^{n!}.$$

The ordinary forms are then

$$e_0 M_k(N,\mathbb{Z}_p) \;,\; e_0 S_k(N,\mathbb{Z}_p).$$

For f an eigenform, f is ordinary if a_p is a p-adic unit

Λ-adic forms:

H Hasse invariant.

$E/_T$ T $\mathbb{F}_p$-scheme

F_{abs} = absolute Frob. $\circlearrowleft E$.

$\omega \in H^0(E, \Omega_{E/T})$ $\cdot \eta$ $F^*\eta = H(E,\omega)\,\eta$.

$H(E, \alpha\omega) = \alpha^{1-p} H(E,\omega)$

$H \in M_{p-1}(1, \mathbb{F}_p)$, $H(E,\omega) \neq 0 \Leftrightarrow E$ ordinary

E char. zero lift of H. some power of H.

$S = X[\frac{1}{E}]$, $S_m = S/(\mathbb{Z}/p^m\mathbb{Z})$

S_1 = ordinary locus of $X/\mathbb{F}_p$.

$$\begin{array}{c} \mathbb{E} \\ \downarrow \\ S_m \end{array} \qquad \mathbb{E}[p]^\circ \underset{\substack{\text{loc.}\\ \text{etale}}}{\simeq} \mu_p \quad , \mathbb{E}[p^m]^\circ \underset{\substack{\text{loc}\\ \text{etale}}}{\simeq} \mu_{p^m}.$$

$$P_n := (\mathbb{E}[p^n]^\circ)^* \underset{\substack{\text{loc}\\ \text{etale}}}{\simeq} (\mathbb{Z}/p^n\mathbb{Z}).$$

Igusa tower:

$$T_{n,m} = \mathrm{Isom}_{/S_m}(P_n, (\mathbb{Z}/p^n\mathbb{Z})).$$

$$\begin{array}{c} T_{n,m} \\ \downarrow \text{etale} \\ S_m \end{array}$$

Thm (Igusa): $T_{n,m}$ irreducible cover of S_m of Galois grp

$$\cong (\mathbb{Z}/p^n\mathbb{Z})^\times.$$

Lemma: $\omega/_{T_{n,m}} \cong \mathcal{O}_{T_{n,m}}$ $\quad n \geq m.$

Proof: $\mathrm{Lie}\, \mathbb{E} = \mathrm{Lie}\, \mathbb{E}[p^n] = \mathrm{Lie}\, \mathbb{E}[p]^\circ.$ $\qquad$ (over $T_{n,m}$)

$$\cong \mathrm{Lie}\, \mu_{p^n} \cong \mathcal{O}_{T_{n,m}} \frac{dx}{x} \cong \mathcal{O}_{T_{n,m}} \qquad \square$$

Corollary: $H^\circ(S_m, \omega^k) = H^\circ\left((\mathbb{Z}/p^n\mathbb{Z})^\times, H^\circ(T_{n,m}, \mathcal{O}_{T_{n,m}})(k)\right).$

$$\underset{x}{\overset{\omega}{}} \qquad x \cdot_k f = x^k (x \cdot f).$$

$V_{n,m} = H^\circ(T_{n,m}, \mathcal{O}_{T_{n,m}})$

Ordinary part:

U_p - operator. $\qquad S_m \xrightarrow{\varphi} S_m$ $\quad$ ← lifting of Frob. $\qquad$ (deg p map)

$$(E, i)_{/T} \longmapsto (E/E[p]^\circ, i).$$

T char p.

$$E \longmapsto E^{(p)}$$

$$\begin{tikzcd}
E \arrow[r] \arrow[rr, bend left] \arrow[dr] & E^{(p)} \arrow[r] \arrow[d] & E \arrow[d] \\
& T \arrow[r, "\mathrm{Frob}"'] & T
\end{tikzcd}$$

$U_p: H^\circ(S_m, \omega^k) \to H^\circ(S_m, c\varphi^* \omega^k) \xrightarrow{\frac{1}{p} tr} H^\circ(S_m, \omega^k)$

$$f(q) = \sum a_n q^n \longmapsto p \cdot \sum a_{np} q^n$$

$$e = \lim_{n \to \infty} U_p^{n!}$$

Can also define U_p acting on $V_{n,m}$.

$$V^{ord} := \varinjlim_n \varinjlim_m e V_{n,m}. \quad \circlearrowleft \mathbb{Z}_p^\times.$$

$$H^0(\mathbb{Z}_p^\times, V^{ord}[k]) = e \cdot H^0(S, \omega^k \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$
$$\qquad\qquad\qquad\qquad \shortparallel$$
$$H^0(S, \omega^k/\mathbb{Z}_p) \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

$$\mathcal{V}^{ord} = (V^{ord})^{*} = \mathrm{Hom}_{\mathbb{Z}_p}(V^{ord}, \mathbb{Q}_p/\mathbb{Z}_p)$$

$$S(N, \underline{\Lambda}) = \mathrm{Hom}_{\underline{\Lambda}}(\mathcal{V}^{ord}, \underline{\Lambda}) \quad \text{where } \underline{\Lambda} = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$$

__Prop:__ (Jochnowitz, Hida) $k \geq 3$,

$$e_0 H^0(X(N), \omega^k/\mathbb{F}_p) = e H^0(S, \omega^k/\mathbb{F}_p).$$

$$\Rightarrow \mathbb{Z}_p^\times = \underbrace{(\mathbb{Z}/p\mathbb{Z})^\times}_{=\Delta} \times \underbrace{(1 + p\mathbb{Z}_p)}_{\ni u \text{ top. gen.}}$$

$$\underline{\Lambda} = \mathbb{Z}_p[\Delta][\![1 + p\mathbb{Z}_p]\!] \cong \mathbb{Z}_p[\Delta][\![T]\!]$$
$$u \longmapsto 1 + T$$

$$\Lambda = \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!] = \mathbb{Z}_p[\![T]\!].$$

ω = Teichmuller char.

$a \in \mathbb{Z}/(p-1)\mathbb{Z}$

$S_a(N,\Lambda)$ = the part of $S(N,\Lambda)$ over which Δ acts by ω^a

$P_k \subset \Lambda$, $P_k = (1+T-u^k)$..

Cor: If $k \geq 3$, $k \equiv a \pmod{p-1}$, then

$$S_a(N,\Lambda) \otimes \Lambda/P_k \xrightarrow{\sim} e_0 S_k(N,\mathbb{Z}_p)$$

and similarly for $M_a(N,\Lambda)$..

Λ-adic q-expansion principle:

$$V^{\mathrm{ord}} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p[\![q]\!] \qquad \text{(usual } q\text{-exp prin.)}$$

$$\Rightarrow \quad M_a(N,\Lambda) \longrightarrow \Lambda[\![q]\!] \quad \text{injective.} \qquad k \geq 3$$

$$\left(\sum a_n q^n\right)^{k} \pmod{P_k} \in \mathbb{Z}_p[\![q]\!]$$

is the q-expansion of some $F_k \in e_0 S_k(N,\mathbb{Z}_p)$.

So everything we get specializes to a classical modular form w/ coefficients in $\mathbb{Z}_p$.

Prop:

$$M_a(N,\Lambda) = \left\{ f \in \Lambda[\![q]\!] \text{ s.t. } f \bmod P_k \text{ is the } q\text{-exp. of some } f_k \in e_0 S_k(N,\mathbb{Z}_p) \text{ for } k \gg 0 \right\}$$

Fundamental exact sequence:

$$0 \to S_a(N,\Lambda) \to M_a(N,\Lambda) \longrightarrow \Lambda^{\mathrm{cusp}(\Gamma_1(N))} \longrightarrow 0 \qquad \text{(is a thm)}.$$

Proof: Sufficient to prove modulo m_Λ.

$$e_0 H^0(X, \omega^k/\mathbb{F}_p) \longrightarrow \mathbb{F}_p^{\mathrm{cusp}}$$

$$\parallel$$

$$e H^0(S, \omega^k/\mathbb{F}_p) \longrightarrow \mathbb{F}_p^{\mathrm{cusp}} = e H^0(\mathrm{cusp}, \omega^k) \longrightarrow e H^1(S, \overset{?}{-}) = 0.$$

Λ-adic Eisenstein series: $\quad k \geq 3.$

$$E_k(q) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \qquad \text{where } \sigma_\ell(n) = \sum_{d|n} d^\ell.$$

$$E_k^{\mathrm{ord}}(q) = E_k(q) - p^{k-1} E_k(q^p)$$

$$= \frac{\zeta^{(p)}(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}^{(p)}(n) q^n \qquad \sigma_\ell^{(p)}(n) = \sum_{\substack{d|n \\ (d,p)=1}} d^\ell.$$

$$e_0 : E_k \longmapsto E_k^{\mathrm{ord}}.$$

Kubota-Leopoldt p-adic L-function:

$a \in \mathbb{Z}/(p-1)\mathbb{Z}$, $a \neq 0$, there exists $\mathcal{L}_a \in \Lambda = \mathbb{Z}_p[[T]]$

$$\mathcal{L}_a(u^k - 1) = \zeta^{(p)}(1-k) \quad \text{if } k \equiv a \pmod{p-1}.$$

$$E_a = \frac{1}{2}\mathcal{L}_a + \sum_{n=1}^{\infty} \sigma_{\Lambda,a}(n) q^n$$

with $\displaystyle \sigma_{\Lambda,a}(n) = \sum_{\substack{d|n \\ (d,p)=1}} \langle d \rangle d^{-1} \omega^a(d)$ where

where

$$d \longmapsto \langle d \rangle = (1+T)^{\frac{\log_p d\, \omega(d)^{-1}}{\log_p u}}$$

$$\cap$$

$$\mathbb{Z}_p^\times \longrightarrow 1 + p\mathbb{Z}_p$$

$$E_a(q) \bmod P_k = E_k^{ord}(q)$$

$$\Rightarrow E_a \in M_a(1, \Lambda).$$

(This is due to Serre 1st example of Λ-adic mod. form.)

One gets an action of $T_\ell, \langle \ell \rangle$ $(\ell \nmid N)$, U_ℓ $(\ell \mid N)$, U_p on $M(N, \Lambda)$.

$$H_a(N, \Lambda) \subset \mathrm{End}_\Lambda(M_a(N, \Lambda)).$$

$$h_a(N, \Lambda) \subset \mathrm{End}_\Lambda(S_a(N, \Lambda)).$$

$$E_a | T = \lambda_{\mathbb{E}_a}(T)\, \mathbb{E}_a.$$

$$\lambda_{\mathbb{E}_a}(T_\ell) = \sigma_{\Lambda, a}(\ell).$$

$$\lambda_{\mathbb{E}_a}(U_p) = 1.$$

<u>Λ-adic Eisenstein ideal:</u>

$$I_a \subset h_a(1, \Lambda)$$

ideal generated by $T_\ell - \lambda_{\mathbb{E}_a}(T_\ell)$ $\forall \ell \neq p$,
$U_p - 1$

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[rr, bend right, two heads] & h_a(1, \Lambda) \arrow[r] & h_a(1, \Lambda)/I_a \end{tikzcd}$$

$\mathcal{E}_a$ - kernel of this map $\subset \Lambda$.

So $\Lambda/\mathcal{E}_a \xrightarrow{\sim} h_a(1, \Lambda)/I_a$

Thm: $\xi_a = (\mathcal{Z}_a)$ in Λ, $a \neq 0$, (a even).

Proof: Fundamental exact sq. gives:

$$0 \to S_a(1,\Lambda) \to M_a(1,\Lambda) \to \Lambda \to 0.$$

$$G \in M(1,\Lambda) \quad \text{s.t.} \quad a_0(G) = 1.$$

$$S_a(1,\Lambda) \ni F = \mathbb{E}_a - \tfrac{1}{2}\mathcal{Z}_a \cdot G \qquad a_0(F) = 0$$

$$F \equiv \mathbb{E}_a \pmod{\mathcal{Z}_a}. \qquad a(1,\mathbb{E}_a) = 1$$

$$\Rightarrow a(1,F) \in \Lambda^\times. \qquad (\mathcal{Z}_a \text{ not a unit})$$

$$h_a \longrightarrow \Lambda/\mathcal{Z}_a$$

$$T \longmapsto \frac{a(1,F|T)}{a(1,F)} \equiv \lambda_{\mathbb{E}_a}(T) \bmod \mathcal{Z}_a$$

$$h_a/I_a \cong \Lambda/\xi_a \twoheadrightarrow \Lambda/\mathcal{Z}_a \Rightarrow \mathcal{Z}_a \mid \xi_a.$$

The other direction:

$$F \in S_a(1,\Lambda), \quad \lambda_F \equiv \lambda_{\mathbb{E}_a} \pmod{\xi_a}.$$

ℓ = prime: Look at: $\dfrac{F(q) - F(q^\ell)}{a(1,F)} \equiv \mathbb{E}_a(q) - \mathbb{E}_a(q^\ell) \bmod \xi_a$. $\square$

Galois representations:

f eigenform $\Rightarrow \rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Z}}_p)$.

$$\operatorname{tr}\rho_f(\underset{\substack{\uparrow \\ \text{geometric} \\ \text{Frob.}}}{\mathrm{Frob}_\ell}) = a(\ell, f). \qquad \ell \nmid Np.$$

$\det \rho_f = \chi_f \, \varepsilon^{1-k}$ ε = cyclotomic char.

pseudo representations: (Wiles)

$T: G \longrightarrow A$ s.t. $T(gg') = T(g'g)$

$T(1) = 2.$

some other conditions

If ρ is a rep $G \to GL_2(A) \rightsquigarrow tr(\rho)$ is a pseudo-rep.

$\exists\ \rho_T : G \to GL_2(A) \longleftarrow\!\!\!\longmapsto$ T is a pseudo rep.

s.t. $T = tr\,\rho_T$ A alg. closed field.

① $\rightsquigarrow$ $\exists\ T: G_{\mathbb{Q}} \longrightarrow h_a(N,\Lambda).$ $F_\Lambda = \mathrm{Frac}(\Lambda)$

② $\rightsquigarrow$ $\exists\ \rho_{T_a} : G_{\mathbb{Q}} \longrightarrow GL_2(h_a(N,\Lambda) \otimes F_\Lambda).$

$T_a(\mathrm{Frob}_\ell) = T_\ell \quad \forall\ \ell \nmid Np.$ (conditions for $\ell \mid N$, but omitting discussion of these).

③ $\rho_{T_a}|_{D_p} \sim \begin{pmatrix} \delta_p & * \\ 0 & * \end{pmatrix}$ D_p = decomp. group.

where δ_p is the unramified character of D_p s.t. $\delta_p(\mathrm{Frob}_p) = U_p$.

(if you use arith Frob the δ_p is in the lower right corner).

$\boxed{N=1}$

$\mathbb{V}$:= space of the rep. ρ_{T_a}

$\cong$

$(h_a \otimes F_\Lambda)^2 \ \circlearrowleft\ \rho_{T_a}(c)$ $\quad \mathbb{V} = \mathbb{V}^{\ominus} \oplus \mathbb{V}^{\oplus}$ action of complex conj.

Take $v^+ \in \mathbb{V}^+$, $\mathbb{V}^+ = (h_a \otimes F_\Lambda) \cdot v_1^+$

$\mathbb{L} = h_a$-lattice generated by $\rho_{T_a}(g)\, v^+$, $g \in G_{\mathbb{Q}}$.

$$\Rightarrow \quad \mathbb{L} \otimes F_\Lambda = \mathbb{V}. \quad (\text{using irred of rep } \rho_{T_a})$$

<u>Lemma</u>: $\mathbb{L} = \mathbb{L}^+ \oplus \mathbb{L}^-$,

$\mathbb{L}^+ = h_a \cdot v^+$, $\mathbb{L}^-$ is faithful h_a - module

<u>Proof</u>: $\rho_{T_a}(g) = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix} \in GL_2(h_a \otimes F_\Lambda)$

$T_a(g) = A(g) + D(g) \in h_a$

$T_a(cg) = -A(g) + D(g) \in h_a.$

$\forall g \in h_a[G_\mathbb{Q}] \Rightarrow D(g) \in h_a$

$\mathbb{L}^+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathbb{L}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rho_{T_a}(h_a[G_\mathbb{Q}]) \cdot v^+.$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rho_{T_a}(g) v^+ = D(g) v^+$ $\square$

$I_a \subset h_a$

$\mathbb{L} \otimes h_a / I_a \longrightarrow$

$tr(\rho_{T_a}(g)) \bmod I_a \equiv 1 + \omega^{-a} \langle \varepsilon \rangle^{-1} \varepsilon(g) \pmod{\mathcal{L}_a}.$

$$\Rightarrow \quad 0 \to \mathbb{L}^- \otimes h_a/I_a \to \mathbb{L} \otimes h_a/I_a \to \Lambda/\mathcal{L}_a \to 0$$

$v^+ \longmapsto 1$ ← shows non-split..

$g \longmapsto B(g) \bmod I_a$

$\ker\left(Z^1(G_\mathbb{Q}, \mathbb{L}^- \otimes h_a/I_a (\omega^{-a} \varepsilon \langle \varepsilon \rangle^{-1})) \to H^1(D_p, \text{---}) \right)$

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$$\left(\mathbb{L}^- \otimes h_a / I_a \right)^* \longrightarrow Sel_{G_{\mathbb{Q}}} (\Lambda^* (\omega^{-a} \varepsilon \langle \varepsilon \rangle^{-1}))$$

$$\Phi \longmapsto [c_\Phi]$$

$c_\Phi(g) \in \Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$

$\lambda \longmapsto \Phi(\lambda B(g))$ gives something split, but we know it is

nonsplit so the map is injective.. this gives a lower bound.

$$h_a/I_a = \Lambda/\mathcal{L}_a.$$

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1. Hida theory for $GU(n,n)$
2. Eisenstein ideal for $n=2$.
3. MC for ordinary elliptic curves.

$G = GU(n,n)$ = similitude group for $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$

K = quad. imag.

p = odd prime that splits in K. , $p = \wp \bar{\wp}$.

$K \subset G(\mathbb{A}_f)$ open compact $K = K_p K^p$ $K_p = G(\mathbb{Z}_p)$.

$S_G(K)/\mathcal{O}_{(\wp)}$ $\mathcal{O}_{(\wp)} = \mathcal{O}_K$ localized at $\wp$

= Shimura variety.

$[(A, \lambda, \phi, \imath)]/_T$ $T = \mathcal{O}_{(\wp)}$ - scheme.

- A abelian scheme over T relative dim $2n$
- λ polarization of degree prime to p.
- ϕ is K level structure
- $\imath : \mathcal{O}_K \longrightarrow \mathrm{End}(A)$
- $\omega_{A/T}$ locally free over T

$\omega_{A/T} = \omega^+_{A/T} \oplus \omega^-_{A/T}$

There are arithmetic compactifications for $S_G(K)/\mathcal{O}_{(\wp)}$ (Fujiwara).

$$\begin{array}{c} \mathcal{G} \\ e \left(\big\downarrow \right. \\ \overline{Sh_G(K)} \end{array} \qquad \underline{\omega} = e^* \Omega_{\mathcal{G}/\overline{Sh_G(K)}}$$

$$\underline{\omega} = \underline{\omega}^+ \oplus \underline{\omega}^-.$$

$\mathcal{T} = \mathrm{Isom}_{\overline{Sh}}(\omega^+ \oplus \omega^-, \mathcal{O}^n_{\overline{Sh_G(K)}} \oplus \mathcal{O}^n_{\overline{Sh_G(K)}})$ get a torsor $H \cong GL_n/_{\mathcal{O}_{(\wp)}} \times GL_n/_{\mathcal{O}_{(\wp)}}$.

Any algebraic rep of $H \rightsquigarrow$ automorphic sheaf.

Weights,

$\underline{k} = (k_{n+1}, \ldots, k_{2n}, k_1, \ldots, k_n)$ $k_1 \geq k_2 \geq \cdots \geq k_{2n}$

$\rho_{(k_1,\dots,k_m)}$ irred alg. rep. of GL_m of height weight $(k_1,\dots,k_m)$.

$$\omega_{\underline{k}} = \mathcal{E}^{H}_{x}\ \underbrace{\rho_{(k_1,\dots,k_m)} \otimes \rho_{(k_{2m},\dots,k_{nm})}}_{\rho_{\underline{k}}}.$$

<u>Automorphic forms:</u>

A $\mathcal{O}_{(p)}$-alg.

$$M_{\underline{k}}(K,A) := H^0(\overline{Sh}_G(K), \omega_{\underline{k}/A}).$$

$A=\mathbb{C}$ $\quad \gamma \in G(\mathbb{Q})$, $z \in \mathbb{H}_m = \{ Z \in M_n(\mathbb{C}) : -i({}^t\bar{Z} - Z) > 0 \}$.

$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$j(\gamma, z) = (CZ+D, {}^t(\bar{C}Z+\bar{D})^{-1})$$

$$f : \mathbb{H}_m \longrightarrow V_{\underline{k}}(\mathbb{C}).$$

$M_k(K,\mathbb{C}) =$

$$G(\mathbb{A}_f)/K_\infty := \coprod_x G(\mathbb{Q})\backslash xK, \mathbb{H}_m).$$

$$f \in M_{\underline{k}}(K,\mathbb{C}) \longleftrightarrow f_x : \mathbb{H}_m \longrightarrow V_{\underline{k}}(\mathbb{C}) \text{ holomorphic.}$$

$$(f_x|_{\underline{k}}\gamma)(z) = \rho_{\underline{k}}(j(\gamma,z)^{-1}) f_x(\gamma z) = f_x$$
$$\forall \gamma \in \Gamma_x = x\Gamma x^{-1} \cap G(\mathbb{Q}).$$

If $\underline{k} = (0,\dots,0,k,\dots,k)$

$\Rightarrow$ this is the space of modular forms of wt k.

If r is an integer $0 \le r \le m$, $x \in G(\mathbb{A}_f)$, $\mathfrak{z} \in \mathbb{H}_{n-r}$

$$(\Phi_x F)(z) = \lim_{t \to +\infty} F(x, \begin{pmatrix} \mathfrak{z} & 0 \\ 0 & it1_r \end{pmatrix})$$

(= Siegel operator)

Cusp forms:

$M_k^0 = S_k(K, \mathbb{C}) := \{ F \mid \Phi_x^r(F) = 0 \text{ for } r \geq 1 \}.$

$\cap$

$M_k^1(K, \mathbb{C}) = \{ F \mid \Phi_x^r(F) = 0 \text{ for } r \geq 2 \}$

$= \{ F \mid \Phi_x^1(F) \text{ is cuspidal } \mathbb{H}_{n-1} \}.$

For example, when $n = 2$, $E_\theta \in M_k^1(K, \mathbb{C})$ for some K.

$G(\mathbb{Q}_p) = GL_{2n}(\mathbb{Q}_p) \times \mathbb{Q}_p^\times.$

$\cup$

$G(\mathbb{Z}_p)$

$\cup$

I_m = Iwahori subgroup of depth m.

$= \{ x \in GL_{2n}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times \text{ s.t. } x \bmod p^m = (\,{}_0\nabla, -) \}.$

$I_m^{(1)}$ with 1's on the diagonal.

$I_m / I_m^{(1)} \simeq (\mathbb{Z}/p^m\mathbb{Z})^{\times 2n}.$

$K,\ K_0(p^n) = K \cap I.$

$\psi : T(\mathbb{Z}_p) \longrightarrow \overline{\mathbb{Q}}_p^\times$ finite order

$\psi_{\underline{k}} = \psi[\underline{k}] \quad [\underline{k}]\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{2n} \end{pmatrix} = \prod t_i^{k_i}.$

$\Lambda_n := \mathbb{Z}_p[[T(\mathbb{Z}_p)]] \qquad \Gamma_n$ = pro-p Sylow of $T(\mathbb{Z}_p)$.

$= \mathbb{Z}_p[\Delta_n][[\Gamma_n]]$

$\Lambda_n = \mathbb{Z}_p[[\Gamma_n]] \simeq \mathbb{Z}_p[[T_1, \ldots, T_{2n}]].$

$\underline{a} = (a_{n+1}, \ldots, a_{2n}, a_1, \ldots, a_n) \in (\mathbb{Z}/(p-1)\mathbb{Z})^{2n}.$

Thm: There exist finite free Λ_n-modules $S(K, \Lambda_n)$, $M^1(K, \Lambda_n)$ s.t.

a) Σ Zariski dense set $\subset \{ \psi_{[\underline{k}]}, \psi, \underline{k} \} \subseteq \mathrm{Hom}(T(\mathbb{Z}_p), \mathbb{C}_p^\times)$

s.t. $S(K, \Lambda_n) \otimes_{\Lambda_n, \psi_{[\underline{k}]}} \overline{\mathbb{Z}}_p \simeq e\, S_k(K_0(p^{n_\psi}), \psi, \overline{\mathbb{Z}}_p).$

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$$M'(K,\Lambda_n) \underset{\Lambda_n, \psi_{[k]}}{\otimes} \overline{\mathbb{Z}}_p \simeq eM'(K_0(p^m), \psi, \overline{\mathbb{Z}}_p).$$

b) $\forall x \in G(\mathbb{A}_f)$,

$$F \longmapsto \Big(\sum_{h \in L_x} c_x(h,F) q^h\Big)_{x \in Set}$$

Λ_n-adic q-expansion principle.

$M'(K,\Lambda_n) = \{ \Lambda_n$-adic q-expansion s.t. mod $(\ker \psi_{[k]})$ is the q-exp. of some ord. form in $M'_k(K_0(p^m), \overline{\mathbb{Z}}_p)$. $\}$

($\psi_{[k]}$ dense).

c) Fundamental exact seq.:

$$0 \to S(K,\Lambda_n) \to M'(K,\Lambda_n) \to \bigoplus_{\substack{\text{set of reps } x \\ \text{of } G(\mathbb{A}_f)/K \\ P^{n-1}(\mathbb{A}_f)}} S(K_x, \Lambda_{n-1}) \otimes_{\Lambda_{n-1}} \Lambda_n \to 0.$$

Eisenstein ideal: $[n=2]$.

$\mathbb{f}$ = Hida family for GL_2. $\chi_{\mathbb{f}}$ the nebentypus.

$\mathbb{f} \in \mathbb{I}[[q]]$, $\mathbb{I}/\mathbb{Z}_p[[W]]$.

$\mathbb{f}$ mod $(1+W-u^k)$ is a form of wt k and neben. $\omega^{k-2}\chi_{\mathbb{f}}$.

$\mathbb{D} = (\mathbb{f}, \psi_0, \xi_0, S)$ $\quad$ S = set of primes $\supseteq$ ramified primes in $\mathbb{f}, \psi_0, \xi_0, K$.

$$\mathbb{D} \rightsquigarrow E_{\mathbb{D},x} = \sum_{h \in L_x} c_x(h, E_{\mathbb{D},x}) q^h \in \Lambda_{\mathbb{D}}[[q^h]].$$

where

$$\Lambda_{\mathbb{D}} = \mathbb{I}[[W^-_{K,p} \times W_{K,p}]]$$

$W_{K,p}$ free pro-p Sylow of $K^\times \backslash \mathbb{A}_K^\times / \widehat{U(p^\infty)}\, \mathbb{C}^\times$.

$W_{K,p} = W^+_{K,p} \times W^-_{K,p}$, $W_{\mathbb{Q},p} = W^+_{K,p}$.

$$\mathcal{R}_{S,p} = \mathcal{C}^\infty_c(G(\mathbb{A}_f^{S \cup \{p\}}) /\!/ K^{S \cup \{p\}}) \otimes \mathcal{U}_p$$

$$\mathcal{U}_p = \langle u_t = ItI \rangle$$

$$t = \begin{pmatrix} p^{t_1} & & \\ & \ddots & \\ & & p^{t_{2n}} \end{pmatrix} \quad t_1 \le \cdots \le t_{2n}$$

$h_S^{ord}(K)$ = image of $R_{S,p}$ in $\mathrm{End}_{\Lambda_n}(S(K,\Lambda_n))$

universal ordinary cuspidal Hecke algebra.

$H_S^{ord}(K)$ same def only using $M'(K,\Lambda_n)$ instead of $S(K,\Lambda_n)$.

$\lambda_{\mathbb{D}} : R_{S,p} \longrightarrow \Lambda_{\mathbb{D}}$.

$T \cdot E_{\mathbb{D}} = \lambda_{\mathbb{D}}(T) E_{\mathbb{D}}$. $\quad h_S^{ord}(K_{\mathbb{D}}) \otimes \Lambda_{\mathbb{D}}$.

Eisenstein ideal $\rightarrow$ ideal of ~~$h_S(K_{\mathbb{D}}) \otimes \Lambda_{\mathbb{D}}$~~ generated by

~~the~~ the $\{T \otimes 1 - 1 \otimes \lambda_{\mathbb{D}}(T)\} = I_{\mathbb{D}}$ $T \in R_{S,p}$.

$$\mathcal{E}_{\mathbb{D}} \rightarrow \lambda_{\mathbb{D}} \longrightarrow \frac{h_S^{ord}(K_{\mathbb{D}}) \otimes \Lambda_{\mathbb{D}}}{I_{\mathbb{D}}}.$$

kernel

$\mathcal{E}_{\mathbb{D}} \subset \Lambda_{\mathbb{D}}$

Want to now study $\mathcal{E}_{\mathbb{D}}$.

$$\Lambda_{\mathbb{D}} = \mathbb{I}[[W^-_{K,p} \times W_{K,p}]] \supset \mathbb{Z}_p[[W^+, W^-, T^+, T^-]]$$

W^+ var $\quad W^-$ var $\quad T^+, T^-$ vars

<u>Thm</u>: Fix $\mathbb{D}$. Assume:

a) $\bar{\rho}_f$ irred. $\qquad$ ~~ξ_0'~~ $\xi_0' = \xi|_{\mathbb{A}_{\mathbb{Q}}^\times}$.

b) $\xi_0 \equiv 1$, $\chi_f \equiv 1$ (modulo max ideal)

$\exists\, \ell$ s.t. $\bar{\rho}_f|_{D_\ell}$ indecomposable. $(\ell \neq p)$

Then: $\mathcal{L} = \mathcal{L}^S_{\chi_f \omega \xi_0'}((1+W^+)^{-1}(1+T_+)^{-1}u^3 - 1) \times \mathcal{L}^S_{f,\xi_0^{-c}}((1+T_+)^{-1}u - 1\xi, T_-)$.

divides the Eisenstein ideal $\mathcal{E}_{\mathbb{D}}$.

<u>Idea of proof</u>: For each $x \in P(\mathbb{A})\backslash G(\mathbb{A})/K$,

$$\Phi_x E_{\mathbb{D}} = \mathcal{L} \times \mathbb{G}_x$$

$$\mathbb{G}_x \in S(K_x, \Lambda_1) \otimes_{\Lambda_1} \Lambda_{\mathbb{D}}.$$

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Thm (c) $\Rightarrow \exists\ G \in M^1(K, \Lambda_2) \otimes \Lambda_{\mathbb{D}}$ s.t.

$\Phi_x G = G_x \ \forall x.$

$\Rightarrow \Phi_x(\underbrace{E_{\mathbb{D}} - \mathcal{L} G}_{F}) = 0 \quad \forall x.$

$\in S(K, \Lambda_2) \otimes \Lambda_{\mathbb{D}}.$

a) & b) and computation of f.c.'s & Vatsal thm $\Rightarrow$

$c_x(F,h) = c_x(E_{\mathbb{D}}, h) \in \Lambda_{\mathbb{D}}^\times$

$E_{\mathbb{D}}$ mod $m_{\Lambda_{\mathbb{D}}} \neq 0$. So can pick a $h \in L_x$ s.t. above,

$\Rightarrow h_s^{ord}(K_{\mathbb{D}}) \otimes \Lambda_{\mathbb{D}} \longrightarrow \Lambda_{\mathbb{D}}/\mathcal{L}$

$$T \otimes 1 \longmapsto \frac{c_x(F|T, h)}{c_x(F, h)} \equiv \lambda_{\mathbb{D}}(T).$$

Application to Selmer groups:

Greenberg style Selmer grp:

$$\begin{array}{c} K_\infty \\ \mathbb{Z}_p^2 \ | \\ K \end{array} \qquad \Gamma_K = Gal(K_\infty/K) \xrightarrow[\text{Artin rec.}]{\sim} W_{K,p}$$

S set of ~~finite~~ primes

$Sel_{K_\infty}(\mathbb{f}, \xi_0) = \ker\big(H^1(K_\infty, \rho_{\mathbb{f}} \otimes \xi_0 \otimes \mathbb{I}^*)$
$\longrightarrow \bigoplus_{\ell \in S \setminus \{p\}} H^1(I_\ell, -) \bigoplus_{\wp | p} H^1(I_\wp, \rho_{\mathbb{f}}/\rho_{\mathbb{f}}^+ \otimes \xi_0 \otimes \mathbb{I}^*)\big)$

$$\rho_{\mathbb{f}}|_{D_p} \sim \begin{pmatrix} \delta_p & * \\ & * \end{pmatrix} \ \rho_{\mathbb{f}}^+.$$

$X^S(\mathbb{f}, \xi_0) =$ Pont. dual of $Sel_{K_\infty}(\mathbb{f}, \xi_0)$.

If $\xi_0 = 1$, Kato $\Rightarrow X^S_{K_\infty}(\mathbb{f}) = X^S_{K_\infty}(\mathbb{f}, 1)$ is torsion over $\mathbb{I}[[W_{K,p}]]$.

By a control Thm.

$$X^S_{\mathbb{Q}_\infty}(\mathbb{f}) \oplus X^S_{\mathbb{Q}_\infty}(\mathbb{f} \otimes (\tfrac{\ }{K/\mathbb{Q}})) \cong X^S_{K_\infty}(\mathbb{f}) \otimes_{\Lambda_K} \Lambda_{\mathbb{Q}}.$$

$$\Lambda_{\mathbb{Q}} = \Lambda_K^+ = \mathbb{Z}_p[[W_{K,p}^+]] = \mathbb{Z}_p[[\underset{\substack{\| \\ Gal(\mathbb{Q}_\infty/\mathbb{Q})}}{W_{\mathbb{Q}_p}}]]$$

(Assume exis. of Galois rep)

<u>Thm:</u> Assume $\xi_0 = 1$, ~~...~~ & cond a) & b).

then

~~... divides the char power~~

$\mathcal{L}^S_{\mathbb{f},2}$ divides the char. power series of $X^S_{K_\infty}(\mathbb{f})$.

<u>Idea of proof:</u> construct a pseudo rep. of G_K.

Φ ~~...~~

$$T : G_K \longrightarrow h^S(K_\mathbb{D}).$$

s.t. $T(g^{\mathrm{Frob}_\ell}) \bmod \mathcal{L} \equiv \lambda_\mathbb{D}(T_\ell) \equiv \Psi^c \varepsilon^{-1}(\mathrm{Frob}_\ell) + \mathrm{tr}\,\rho_\mathbb{f}(\mathrm{Frob}_\ell)\xi^{-1}\varepsilon^{-2}\Psi^{-c}(\mathrm{Frob}_\ell)$

$\ell \notin S \cup \{p\}$ $\qquad \oplus\, \Psi^c \xi^{-1} \det \rho_\mathbb{f}\, \varepsilon^{-1}(\mathrm{Frob}_\ell)$

$\Rightarrow$ construct a lattice in $(h^S(K_\mathbb{D}) \otimes \mathrm{Frac}(\Lambda_\mathbb{D}))^{\sim}$ s.t.

$$\bmod \mathcal{L} \quad \begin{pmatrix} \Psi^c \xi^{-1} \det\rho_\mathbb{f}\, \varepsilon^{-1} & * & \boxed{*} \\ * & \rho_\mathbb{f} \otimes \Psi^c \xi^{-1} \varepsilon^{-2} & * \\ & & \Psi^c \varepsilon^{-3} \end{pmatrix}$$

ordinary cond & polarization $\rho^c = \rho^\vee \otimes \xi$.

$\to$ elt in $\mathrm{Sel}_\mathbb{Q}(\xi^{-1} \det\rho_\mathbb{f}\, \varepsilon^{-2})$

$\to$ elt in $\mathrm{Sel}_K(\rho_\mathbb{f} \otimes \varepsilon \xi^{-c}) = \mathrm{Sel}_{K_\infty}(\rho_\mathbb{f} \otimes \varepsilon \xi_0^{-c})$