Skinner
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Iwasawa Theory.

p-adic L-functions

$\updownarrow$

Selmer groups

Eisenstein series : ① construct p-adic L-functions (Eisenstein measure)

② relate congruences of cusp forms and E.S.'s to L-values (Eisenstein ideal).

Kubota-Leopold p-adic L-function:

$$q = \begin{cases} p & p > 2 \\ 4 & p = 2. \end{cases}$$

χ Dir. char. of cond $\mid Np$, $p \nmid N$, takes values in $\mathcal{O}$:= fin. ext. of $\mathbb{Z}_p$.
= p-adic integer ring

$\exists$ $h = h_\chi \in \mathcal{O}[[T]]$ s.t. if

$$f = \begin{cases} h_\chi & \text{if } \chi \neq 1 \\ h_\chi / \left(1 - \frac{1+q}{1+T}\right) & \text{if } \chi = 1. \end{cases}$$

Then

$$L_p(\chi\psi_\zeta, s) := f(\zeta(1+q)^s - 1)$$

satisfies

$$L_p(\chi\psi_\zeta, 1-k) = (1 - \chi\psi_\zeta\omega^{-k}(p)p^{k-1})\, L(1-k, \chi\psi_\zeta\omega^{-k})$$

Teichmuller char.

$k \geq 1$

ψ_ζ is a char. of cond. a power of p order $\left((\mathbb{Z}/p^n\mathbb{Z})^\times \simeq \mu_{p-1} \times (1+p\mathbb{Z}_p)/1+p^n\mathbb{Z}_p \right)$

p-odd

$\psi_\zeta(1+q) = \zeta = p^{th}$ power root of unity measures:

X = compact totally disconnected space (ex. $X = \mathbb{Z}_p$).

R = p-adically complete ring (ex: $R = \mathbb{Z}_p$).

$C(X, R)$ = continuous maps from X to R.

$= C(X, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} R$.

$$\left.\begin{array}{ll} \mu : C(X,R) \to R & R\text{-linear} \\ (C(X,\mathbb{Z}_p) \to R & \mathbb{Z}_p\text{-linear}) \end{array}\right\} \text{measure}$$

$$\int_X f\, d\mu := \mu(f).$$

$$\begin{array}{l} (1+q)^a \quad \Gamma = 1+q\,\mathbb{Z}_p \leq \mathbb{Z}_p^\times . \\ \quad\;\; \uparrow \wr \\ a \quad\;\; \mathbb{Z}_p \end{array}$$

$$\Lambda = \mathbb{Z}_p[\![\Gamma]\!] = \varprojlim_n \mathbb{Z}_p[\Gamma/_{1+qp^n\mathbb{Z}_p}] \xrightarrow{\sim} \mathbb{Z}_p[\![T]\!]$$

$$1+q \in \Gamma \longmapsto 1+T.$$

$\mathcal{O}$-valued measures on Γ $\longleftrightarrow$ elements in Λ

$\mu_f \longleftarrow f$

$$\int \psi_\zeta\, x^s d\mu_f = f(\zeta(1+q)^s - 1).$$

So one can think of p-adic L-functions as closed related to measures.

Construction of p-adic L-functions:

ψ Dirichlet char. mod M.

$\psi(-1) = (-1)^k \quad k > 0$

$$E_k(\psi, z) = \underbrace{\frac{L(1-k,\psi)}{2}}_{a_k(0,\psi)} + \sum_{n=1}^{\infty} \underbrace{\left(\sum_{d|n} \psi(d) d^{k-1}\right)}_{c_k(n,\psi)} q^n \qquad q = e^{2\pi i z}.$$

χ as before

__Example:__ $a(\chi, \ell : T) = \chi(\ell)\, \ell^{-1} \langle \ell \rangle + 1 \quad \in \mathcal{O}[[T]]$.

$\ell \neq p \qquad \langle \ell \rangle = (1+T)^{c_\ell} \qquad ((1+q)^{c_\ell} \omega(\ell) = \ell)$

$$a(\chi, \ell, \zeta(1+q)^k) = \chi(\ell)\, \ell^{-1} \psi_\zeta(\ell) (1+q)^{kc_\ell}$$

$$= \chi \psi_\zeta \omega^{-k}(\ell)\, \ell^{k-1} + 1.$$

So we see these Eisenstein series fit into nice p-adic families.

Serre (Antwerp III) observed that if $\{f_i = \sum_{n=0}^{\infty} a_i(n) q^n\}$ w/ p-adic Fourier coefficient and $\forall n > 0$, $\{a_i(n)\}$ converges, then so does $\{a_i(0)\}$.

In particular, there exists a p-adic analytic function (at least if $\chi \neq 1$)

there is an analytic function $a(\chi, 0; T)$ s.t.

$$a(\chi, 0; \zeta(1+p)^k - 1) = L_p(\chi \psi_\zeta, 1-k).$$

__Katz:__ basic idea

$$\tilde{E}_k(\chi, z) = E_k(\chi, z) - E_k(\chi, \ell z) = \sum_{n=1}^{\infty} c_k(\chi, n) q^n$$

$C_k(\chi \psi_\zeta \omega^{-k}, n)$'s - values of measures on Γ.

With work one can find a measure μ_E taking values in the space of p-adic modular forms with p-integral Fourier coeffs at cusps unramified at p.

$$\int \Psi_\xi x^k \, d\mu_E = \tilde{E}_k(\chi w^k \Psi_\xi, z).$$

Recover the p-adic L-function from the measure giving the constant term at another cusp (ramified at ℓ).

<u>Remark:</u> Deligne and Ribet followed this ay. to construct p-adic L-functions for Hecke char's of totally real fields.

K = imaginary quadratic field.

$$K^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Q}} K \simeq GL_2$$

$$E_k(\Psi, z) \longrightarrow E_k(\Psi, g) \text{ on } GL_2(\mathbb{A})$$

∞-type of $w_E = \Psi_\infty |\cdot|^{k-2}$ or $2-k$.

$$= j(g,i)^{-k} \det g^{k-1} E_k(\Psi, g(i)).$$

<u>Key Formula:</u>

$$\int_{K^\times \backslash \mathbb{A}_K^\times / \mathbb{A}_{\mathbb{Q}}^\times} E_k(\Psi, t)\chi(t)\, d^\times t = (*)\, L^S(k-2, \chi)$$

$$\chi|_{\mathbb{A}_{\mathbb{Q}}^\times} = w_E^{-1} \text{ depends on } \Psi \text{ and } k.$$

$$K^\times \backslash \mathbb{A}_K^\times / \mathbb{A}_{\mathbb{Q}}^\times \longrightarrow GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / SO_2(\mathbb{R})\, \mathbb{R}_{>0}\, K_0(N).$$

image is a set of CM points

So we can reinterpret the key formula as a weighted sum (weights depend on χ)

of $E_K(\Psi, z)$ over certain CM points.

$$E_K(\Psi, z) \text{ - functions on } (\overset{\swarrow \text{elliptic curve}}{E}, \eta, \omega)$$

So then CM points correspond to CM elliptic curves.

The values of the Eisenstein series are (CM periods) $\times$ $\left(\begin{smallmatrix}\text{something}\\ p\text{-integral}\end{smallmatrix}\right)$

Can construct a measure (two variable) interpolating the $L(\kappa-2, \chi)$.

<u>Remark:</u> The key formula is a special case of "doubling" for unitary groups.

K : $\langle a, b\rangle = a\bar{b}$. unitary group $U(1)$.

$K \oplus K$ has pairing $\langle\langle\ ,\ \rangle\rangle = \langle\ ,\ \rangle \oplus -\langle\ ,\ \rangle$. sig $(1,1)$

unitary group is $U(1,1) \supseteq SL_2$

$$\begin{array}{ccc} & & E_K(\Psi) \\ U(1)\times U(1) & \hookrightarrow & U(1,1) \\ & & E_K(\Psi). \end{array}$$

$$\int_{U(1)_{\mathbb{Q}} \times U(1)_{\mathbb{Q}} \backslash U(1)_{\mathbb{A}} \times U(1)_{\mathbb{A}}} E_K(t_1, t_2)\, \chi(t_1)\chi'(t_2)\, dk = (*)\, L^S(\kappa-2, \chi) \langle \chi, \chi' \rangle.$$

<u>Rankin-Selberg:</u>

$f \in S_k(N, \chi)$ $\qquad f = \sum_{n=1}^{\infty} a(n) q^n$

$g \in M_\ell(N, \Psi)$ $\qquad g = \sum_{n=0}^{\infty} b(n) q^n$ $\qquad \ell < k$.

$$\int_{\Gamma_0(N)\backslash \mathfrak{H}} f\, \overline{E_{k-\ell}(\chi\Psi^{-1}, z) g}\, y^k\, d\text{vol}. = (*)\, L(f \times \bar{g}, k-1)$$
$$\shortparallel$$
$$\sum a(n)\overline{b(n)}\, n^{-s}.$$

Can vary f, g, E in p-adic families.

Can construct the p-adic L-function of an eigenform f this way.

Take g to be an Eisenstein series, $g = E_\ell(\mu, \lambda)$, L-function of g $L(\mu, s) L(\lambda, s-\ell+1)$. Then

$$L(f \times \bar{g}, k-1) = L(f \times \mu^{-1}, k-1)\, L(f \times \lambda^{-1}, k-\ell).$$

- Constructing p-adic L-functions
- Construct p-adic families of cuspidal Eisenstein series & understand the constant terms and Fourier coefficients.

This is all going to be in the context of unitary groups.

Unitary groups:

K = imaginary quadratic field.

V = n-dim. K-space , $\langle\,,\rangle_V : V \times V \to K$ skew Hermitian

$\left(ex. \begin{pmatrix} & 1_m \\ -1_m & \end{pmatrix} \quad n = 2m \right)$

$G = U(V)$

$W = V \oplus K^{2r}$, $\langle\,,\rangle_W = \langle\,,\rangle_V \oplus \begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}$ ← pairing assoc to this matrix.

$r \geq 0$

$H = U(W)$.

$H \supset P = \mathrm{stab}_H(\{ \underset{V}{0} \oplus \underset{K^r}{0} \oplus x : x \in K^r \})$

$P = MN$, $\quad M \simeq U(V) \times GL_r(K)$

$m(g,a) \longleftrightarrow (g, a)$

$\begin{pmatrix} g & & \\ & {}^t\bar{a}^{-1} & \\ & & a \end{pmatrix}$

$\mathbb{V} = W \oplus V$, $\langle\langle\,,\rangle\rangle = \langle\,,\rangle_W \oplus -\langle\,,\rangle_V$ This has signature $(n+r, n+r)$.

$\mathbb{G} = U(\mathbb{V})$

$\cup$

$H \times G$

$\mathbb{V} \supseteq \mathbb{X} = \{ (\underbrace{v \oplus 0 \oplus x}_{\in W} \oplus v) \quad v \in V, x \in K^r \}$

$\mathbb{P} = \mathrm{stab}_{\mathbb{G}} \mathbb{X}$

$= \mathbb{M}\,\mathbb{N}$, $\quad \mathbb{M} \simeq \mathrm{Aut}_K \mathbb{X} \simeq GL_{n+r}(K)$

$$\mathbb{P} \cap H\times G = \{(h,g) : h = m(g,a)n\}.$$

Cuspidal E.S:

(π, V_π) cuspidal auto. rep of $G(\mathbb{A})$. $(\mathbb{A} = \mathbb{A}_{\mathbb{Q}})$

χ - Hecke character of K. (compose w/ det. to get a char of $GL_r(K)$).

Together these give a rep of $M(\mathbb{A})$, extend it to $P(\mathbb{A})$.

$s \overset{f}{\mapsto} f_s \in \mathrm{Ind}_P^H(\pi \otimes \chi\delta_P^s)$ meromorphic sections, can think of as auto forms on the parabolic.

$$E(f,s,h) = \sum_{\gamma \in P(\mathbb{Q})\backslash H(\mathbb{Q})} f_s(\gamma h)$$ · converges for $\mathrm{Re}(s) \gg 0$
· mero cont..

$\underline{r=1}$ only nonzero constant term is along P

$$E(f;s,h)_P = f_s(h) + \underbrace{M(s)}_{} f_s(h)$$

related to L-values

$$\frac{L^S(\pi,\chi^{-1},(n+1)s)\, L^S((\chi^{-1})|_{\mathbb{A}_{\mathbb{Q}}}, 2(n+1)s)}{L^S(\pi,\chi^{-1},(n+1)s+1)\, L^S((\chi^{-1})|_{\mathbb{A}_{\mathbb{Q}}}, 2(n+1)s+1)}$$

(Langlands normalization $s \to 1-s$).

Remarks: (a) If π_∞ is a holo. discrete series & χ_∞ alg. can often choose $f_{\infty,s}$ so that

$$E(f,s_0,h) \longrightarrow$$ holo. form on symm. space assoc to H.

(b) to "do arithmetic" need:

good sections at all places

- understand $M(s)f_s$
- Hecke operators
- compute F.C.'s.

Example: $n=2$, $r=1$, $\langle\,,\,\rangle_V = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

symm. domain assoc. to G is the usual upper half plane.

Can choose a basis of W so that $\langle\,,\,\rangle_W = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$. The

symm. domain assoc to H is $\mathbb{H}_2 = \{ Z \in M_2(\mathbb{C}) : -i(Z - {}^t\bar{Z}) > 0 \}$.

Eisenstein data:

$\varpi = (f, \psi, \xi)$ $\quad$ $f =$ holo cuspidal e.f. (on $\mathbb{H}$) of wt k)

$\psi =$ Hecke char. of K, s.t. $\psi((m)) = \chi_f(m) m^k$.

$\xi =$ finite order char of K.

$f \longrightarrow f_{/\mathbb{A}}$

$\psi \longrightarrow \psi_{/\mathbb{A}}$ $\quad \psi_\infty = \left(\frac{z}{|z|}\right)^{-k}$

$\xi \longrightarrow \xi_{/\mathbb{A}}$

adelic & unitary

$G(\mathbb{A}) \subseteq \mathbb{A}_K^\times GL_2(\mathbb{A}) \subseteq GL_2(\mathbb{A}_K)$

$\quad\; \| \qquad\qquad\qquad \|$

$U(1,1)_{/\mathbb{A}} \qquad GU(1,1)_{/\mathbb{A}}$

so $f_{/\mathbb{A}}$ & $\psi_{/\mathbb{A}}$ $\rightsquigarrow$ cusp form on $G(\mathbb{A})$, generate some π.

for $x \in H(\mathbb{A}_f)$

$$E_{\varpi,x}(Z) = \det(c_h i + d_h)^k E_\varpi(f_\varpi; s_0, hx)$$

$$h(i) = Z, \quad h \in H(\mathbb{R})$$

$$h = \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix}.$$

- f ordinary at p, $E_{\varpi,x}$ is also ordinary ($\chi_p = 1$).
- $E_{\varpi,x}(Z) = \sum_{\substack{T \geq 0 \\ T \in M_2(K) \\ {}^t\bar{T} = T}} c_{T,x}\, e(\mathrm{tr}(TZ))$.

$\xi' = \xi|_{\mathbb{A}_\mathbb{Q}}$

$\det T = 0$, $T \neq 0$: $c_T = * \; L^{\Sigma, alg}(f, \xi, k-1)\, L^{\Sigma, alg}(\chi_f \xi', k-2) \times$ f.c. of f.

(arithmetic normalization)

Aside: Langlands' normalization would be:

$$L(\pi_f \otimes \xi_{/\mathbb{A}} \Psi^c_{/\mathbb{A}}, s)\, L(\Psi^c_{/\mathbb{A}}, s+\tfrac{k-3}{2})\, L(\Psi^c_{/\mathbb{A}} \chi_f \xi'_{/\mathbb{A}}, s-\tfrac{k-3}{2})$$

base change to K, then twist...

Siegel E.S. on $\mathbb{G}$ and pullbacks to $H \times G$:

χ - Hecke char of K $\rightsquigarrow$ function on $\mathbb{M}(\mathbb{A})$ by composing w/ det.

$s \xrightarrow{f} f_s \in \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \delta_P^s$

$$\mathbb{E}(f; s, g) = \sum_{\gamma \in \mathbb{P}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{Q})} f_s(\gamma g).$$

- $\chi_\infty(z) = \left(\frac{z}{|z|}\right)^{-k}$ can choose $f_{\infty,s}$ so that

  $\mathbb{E}(f; s_0, g) \longrightarrow$ holo. of wt k on $\mathbb{H}_{n+r}$. (usually)

- Can explicitly compute the F.C's of $\mathbb{E}$'s. (Shimura, Garrett).

$$(*) \quad \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \mathbb{E}(f; s, (g,h))\, \bar{\chi}(\det g)\, \varphi(g)\, dg \qquad \varphi \in V_\pi$$

unwind the $\mathbb{E}$ by considering the orbits of $H(\mathbb{Q}) \times G(\mathbb{Q})$ on $\mathbb{P}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{Q})$. Turns out there is only one orbit that is not negligible, and obtain

$$= \sum_{\gamma \in P(\mathbb{Q}) \backslash H(\mathbb{Q})} \underbrace{\int_{G(\mathbb{A})} f_s((\gamma h, g))\, \bar{\chi}(\det g)\, \varphi(g)\, dg}_{F_s(\gamma h)}$$

$F_s(h) \in \mathrm{Ind}_P^H(\pi \otimes \chi \delta_P^s)$, so this is a ~~[illegible]~~ cuspidal Eisenstein series.

This is just the adelic interpretation of Shimura's book.

$\underline{r=0}$ this is the doubling method

Problems: (back to the example)

Need to choose $f_s \in \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}}(\chi\delta_{\mathbb{P}}^{s})$ so that corresponding $F_s = f_{\mathcal{B}}$.

and still be able to compute F.C's of $\mathbb{E}(f; s_0, g)$.

(back to example still):

$r=0$: $\quad s_0 = \frac{k-2}{4} \qquad \chi = \psi/\xi$

$$F_{s_0} = (*)\ L^{\Sigma, alg}(f, \xi, k-1) \times$$ ~~[illegible]~~ suitable cusp form f'.

$r=1$: $\quad s_0 = \frac{k-3}{6}$

$$F_{s_0} = (*)\ L^{\Sigma, alg}(f, \xi, k-1) \times \text{section}.$$

p-adic L-functions: $\qquad p$ splits in K.

It is easiest here to take $r=0$.

integral. (inner prod) Shimura formula

What is $(*)$ telling us?

$$\mathbb{E}((h,g)) = \sum_{\substack{\varphi, \varphi' \\ \varphi \in \pi \\ \varphi' \in \tilde{\pi}}} c_{\varphi,\varphi'}\, \varphi \otimes \varphi'.$$

$$\langle \mathbb{E}(h,g), \varphi(g) \rangle_G^{Pet.} = c_{\varphi,\varphi'} \langle \varphi, \varphi \rangle \varphi'.$$

$\shortparallel$

L-value

So $$\frac{\langle \mathbb{E}(h,g), \varphi(g) \rangle_G^{pet.}}{\text{period}} = \left(\frac{c_{\varphi,\varphi'} \langle \varphi, \varphi \rangle}{\text{period}} \right) \varphi'.$$

$\shortparallel$

L-value.

joint (ongoing) work w/ J.S. Li & M. Harris.

Skinner

p-adic families of cuspidal E.S. (example):

$\mathbb{D} = (\mathbb{f}, \psi_0, \xi_0)$ $\mathbb{f} \in \mathbb{I}[[q]]$ $\mathbb{I}$ finite ext into $\mathbb{Z}_p[[W]]$.

Hida family

$\psi_0|_{\mathbb{A}_\mathbb{Q}^\times} = \chi_{\mathbb{f}}$, ξ_0 } finite order.

$W_{K,p}$ free part of $K^\times \backslash \mathbb{A}_K^\times / \mathbb{C}^\times \widehat{U(p^\infty)} \simeq \mathbb{Z}_p^2$ as $\mathbb{Z}_p$-modules.

C.C. acts

$W_{K,p}^\pm \ni \gamma_\pm$, $\Psi_K : \mathbb{A}_K^\times \twoheadrightarrow W_{K,p}$

$\Psi_K^\pm$.

$\exists\ \mathcal{L}_{\mathbb{f},\xi_0} \in \mathbb{I}[[W_{K,p}]]$ s.t. $P \in \mathbb{I}[[W_{K,p}]]$ prime of dim 1.

$P \ni 1 + W - \zeta(1+p)^k$ variable

$P \ni \gamma_\pm - \zeta_\pm$ p-power root of unity.

Then

$$\mathcal{L}_{\mathbb{f},\xi_0} \bmod P = (*)\, L^{\Sigma,\mathrm{alg}}(f_P \times \xi_0 \xi_P, k-1)$$

$$\xi_P = \Psi_K \bmod P$$

[We need that $\bar{\rho}_{\mathbb{f}}$ is irred for this stuff...]

There exists $\mathcal{E}_{\mathbb{D},x} \in \underbrace{\mathbb{I}[[W_{K,p}^- \times W_{K,p}]]}_{\Lambda_\mathbb{D}}[[q^T]]$ $T \leftarrow$ herm. matrices.

P as above, $P \subseteq \mathbb{I}[[W_{K,p}^- \times W_{K,p}]]$.

$P \ni \tilde{\gamma}_- - \tilde{\zeta}$

$\mathcal{E}_{\mathbb{D},x} \bmod P = E_{\varpi_P,x}$

$E_{\varpi_P,x} = (f_P, \psi_0 \psi_P^+ \psi_P^-, \xi_0 \xi_P)$

$\psi_P^- = \Psi_{K,P}^- \bmod P$

$W_{K,p}^-$.

$$\mathcal{E}_{\mathbb{D},\chi} = \sum c_{T,\chi}\, q^T$$

$T \neq 0$, $\det T = 0$ $\quad c_{T,\chi} \in \mathcal{L}_{\mathfrak{f},\xi_0}\, \mathcal{L}_{\chi_f \xi_0}\, \Lambda_{\mathbb{D}}$.

$T > 0$? Need to show at least one of these coefficients is not divisible by the prime dividing the p-adic L-function so that the E.S. doesn't actually vanish mod p.