Tamagawa Number Conjecture & Iwasawa Main conjecture:

Aim: $\zeta_F(0)^* = -\frac{R\,h_F}{w}$ (class number formula)

Dedekind zeta fctn.

We would like to generalize this to motives.

I. TNC (Tamagawa Number conjecture):

I.1 (Artin)-motives:

E a number field. $\mathcal{M}_{\mathbb{Q}}(E)$ category of Chow motives $/\mathbb{Q}$ with coefficients in E. The objects in this category are triples $(X, q, r) = M$ where $X/\mathbb{Q}$ is a smooth proj. variety, $r \in \mathbb{Z}$, and q is an idempotent in $CH(X \times_{\mathbb{Q}} X)_E$.

Dual: $M^\vee := (X, {}^t q, \dim X - r)$ X has pure dim.

Tate Twist: $M(n) := (X, q, r+n)$.

Realizations: $M_? := \bigoplus_{i \in \mathbb{Z}} q^* H^i_?(X, r)_E$. for any twisted cohom. theory.

Example: $E/\mathbb{Q}$ be an elliptic curve. $h(E) = (E, \Delta, 0)$. (field is $\mathbb{Q}$ here)

$$h(E) = h^0(E) + h^1(E) + h^2(E).$$

$$h^1(E)_B := H^1_B(E(\mathbb{C}), \mathbb{Q})$$

$$h^1(E)_{DR} := H^1_{DR}(E/\mathbb{Q}).$$

$$h^1(E)_{et} := H^1_{et}(E \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$$ (This has an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$).

Def: An Artin-motive is a direct sum of motives of the form $(\operatorname{Spec} F, q, 0)$ where $F/\mathbb{Q}$ is a number field.

Notation: $h^0(F) := (\operatorname{Spec} F, \Delta, 0)$.

Example: ① $F = \mathbb{Q}(\zeta_N)$, E = field containing the values of all chars. $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Then we have

$$h^\circ(\mathbb{Q}(\zeta_N))_E = \bigoplus_{\chi:(\mathbb{Z}/N\mathbb{Z})^\times \to E^\times} V(\chi)$$

② Realizations of $h^\circ(F)(r) = (\operatorname{Spec} F, \Delta, r)$, $E = \mathbb{Q}$.

$$h^\circ(F)(r)_{DR} = F$$

$$h^\circ(F)(r)_B = \bigoplus_{\sigma: F \to \mathbb{C}} (2\pi i)^r \mathbb{Q}.$$

$$h^\circ(F)_p = h^\circ(F)(r)_{ét} = H^\circ_{ét}(F \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p(r)).$$

I.2 L-functions:

Def: Let M be an (Artin) - motive. Define

$$D_\ell(M_p) := \begin{cases} M_p^{I_\ell} & \ell \neq p \quad I_\ell = \text{inertia} \\ (B_{cris} \otimes_{\mathbb{Q}_p} M_p)^{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} & \ell = p \end{cases}$$

with an action of (geometric) Frob. Fr_ℓ.

Example: $D_p(h^\circ(F)_p) = \bigoplus_{\wp | p} F_\wp^\circ$ where $F_\wp^\circ \subset F_\wp$ is the maximal unramified subfield.

Def: Let ~~[illegible]~~ $P_\ell(M_p, t) = \det(1 - Fr_\ell t \mid D_\ell(M_p))$.
M has an L-function if $P_\ell(M_p, t) \in E[t]$ and $L(M,s) := \prod_\ell P_\ell(M_p, \ell^{-s})^{-1}$ converges for $s \gg 0$. Note $L(M,s) \in \mathbb{C} \otimes_{\mathbb{Q}} E$.

Example: ① $L(h^\circ(F)(r)_p, s) = \zeta_F(s+r)$

② $L(V(\chi), s) = \left(\sum_{n>0} \frac{\tau \circ \chi(n)}{n^s} \right)_{\tau: E \to \mathbb{C}}$

<u>Conjecture 1:</u> M has an L-function which admits meromorphic continuation into a nbd of 0.

<u>Note:</u> $L(M(n),s) = L(M, s+n)$

<u>Def:</u> Assume conj. 1. Let $d_M := \operatorname{ord}_{s=0} L(M,s)$. Define $L(M,0)^* := \lim_{s\to 0} \frac{L(M,s)}{s^{d_M}} \overset{!}{\in} (E\otimes_{\mathbb{Q}}\mathbb{R})^\times$ leading term of Taylor series expansion.

I.3 Motivic and unramified cohomology:

<u>Def:</u> $M = (X,q,r)$. Let $H^0_{\mathcal{M}}(\mathbb{Q},M) := q^* CH^r(X)_E / q^* CH^r(X)^0_E$.

and $H^1_{\mathcal{M}}(\mathbb{Q},M) := q^* CH^r(X)_E \oplus \bigoplus_{\substack{i\in\mathbb{Z}\\ i\neq -1}} q^*(K_{-i-1}(X)\otimes_{\mathbb{Z}} E)^{(r)}$.

where $CH^r(X)_E$ r^{th} Chow group with coefficients in E

$CH^r(X)^0_E \subset CH^r(X)_E$ cycles homo. ~ 0 and

$K_*(X)$ K-theory of X. and the superscript (r) is the r^{th} adams eigenspace.

<u>Example:</u> ① $H^0_{\mathcal{M}}(\mathbb{Q}, h^0(F)(r)) = \begin{cases} 0 & \text{if } r\neq 0 \\ \mathbb{Q} & \text{if } r=0 \end{cases}$

② $H^0_{\mathcal{M}}(\mathbb{Q}, V(\chi)) = \begin{cases} 0 & \chi\neq 1 \\ E & \chi = 1. \end{cases}$

③ $H^1_{\mathcal{M}}(\mathbb{Q}, h^0(F)(r)) = \begin{cases} F^\times\otimes\mathbb{Q} & r=1 \\ K_{2r-1}(F)\otimes\mathbb{Q} & r>1 \\ 0 & r\leq 0. \end{cases}$

Soulé regulator & Abel-Jacobi map $\rightsquigarrow$

$$r_p : H^1_{\mathcal{M}}(\mathbb{Q}, M) \longrightarrow H^1_{cont}(\mathbb{Q}, M_p).$$

<u>Example:</u> $M = h^0(F)(r)$

$$r_p : K_{2r-1}(F) \otimes \mathbb{Q} \longrightarrow H^1_{cont}(F, \mathbb{Q}_p(r)).$$

<u>Def:</u> Complex of unram (loc) cohomology classes

$$R\Gamma_f(\mathbb{Q}_\ell, M_p) := \begin{cases} D_\ell(M_p) \xrightarrow{Fr_\ell - 1} D_\ell(M_p) & \ell \neq p \\ D_\ell(M_p) \xrightarrow{(Fr_\ell - 1,\ can)} D_p(M_p) \oplus t_{M_p} & \ell = p \end{cases}$$

with

$$t_{M_p} := \left(B_{dR}/\mathrm{Fil}^0 B_{dR} \otimes M_p \right)^{Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}.$$

One has $\alpha : R\Gamma_f(\mathbb{Q}_\ell, M_p) \longrightarrow R\Gamma(\mathbb{Q}_\ell, M_p)$

<u>Def:</u> $R\Gamma_{/f}(\mathbb{Q}_\ell, M_p) = \mathrm{cone}(\alpha)$

<u>Example:</u> $M_p = h^0(F)(1)_p$ then

$$0 \to H^1_f(\mathbb{Q}_p, M_p) \to H^1(\mathbb{Q}_p, M_p) \to H^1_{/f}(\mathbb{Q}_p, M_p) \to 0$$

$$\bigoplus_{\mathfrak{g}|p} \mathcal{O}_{F_\mathfrak{p}}^{*\wedge} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \hookrightarrow \bigoplus_{\mathfrak{g}|p} F_\mathfrak{g}^{*\wedge} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{val} \bigoplus_{\mathfrak{g}|p} \mathbb{Q}_p.$$

$\wedge$ = p-adic completion

<u>Def:</u> $H^1_f(\mathbb{Q}, M_p) := \ker\left(H^1_{cont}(\mathbb{Q}, M_p) \longrightarrow \bigoplus_\ell H^1_{/f}(\mathbb{Q}_\ell, M_p)\right)$

This is the <u>Selmer group.</u>

$$H^1_{\mathcal{M}}(\mathbb{Z}, M) := \left\{ x \in H^1_{\mathcal{M}}(\mathbb{Q}, M) \;\middle|\; r_p(x) \in H^1_f(\mathbb{Q}, M_p) \ \forall p \right\}$$

This is the "integral" motivic cohomology.

<u>Example:</u> $M = h^0(F)(r)$ $\quad H^1_\mu(\mathbb{Z}, M) = \begin{cases} \mathcal{O}_F^\times \otimes \mathbb{Q} & r=1 \\ K_{2r-1}(\mathcal{O}_F) & r>1. \end{cases}$

<u>Conj. 2:</u> a) $H^0_\mu(\mathbb{Q}, M)$ and $H^1_\mu(\mathbb{Z}, M)$ are f.d. E v.s's.

b) cycle class: $H^0_\mu(\mathbb{Q}, M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} H^0(\mathbb{Q}, M_p)$

(Tate-conjecture reformulated).

c) $r_p : H^1_\mu(\mathbb{Z}, M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} H^1_f(\mathbb{Q}, M_p) \quad \forall p.$

d) $d_M = \operatorname{ord}_{s=0} L(M,s) = \dim_E H^1_\mu(\mathbb{Z}, M^\vee(1)) - \dim_E H^0_\mu(\mathbb{Q}, M^\vee(1)).$

<u>Thm (Borel, Soulé):</u> Quillen
If M is an Artin motive, then conj. 2 is true.

<u>Example:</u> $\operatorname{ord}_{s=0} L(h^0(F), s) = \dim_{\mathbb{Q}} H^1_\mu(\mathbb{Z}, h^0(F)(1)) - \underbrace{\dim_{\mathbb{Q}} H^0(\mathbb{Q}, h^0(F)(1))}_{=0}$

$= \dim_{\mathbb{Q}}(\mathcal{O}_F^\times \otimes \mathbb{Q}).$

as it should be.

Let $I_\infty : M_B \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}.$

$\rightsquigarrow \alpha_M : \underset{\substack{\uparrow \\ \text{inv under} \\ F_\infty \circ c}}{M_B^+} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \underbrace{M_{dR}/F^0 M_{dR}}_{=: \tan_M} \otimes \mathbb{R}.$

This is the period map.

Connection to H_μ:

<u>Conj 3:</u> The Beilinson regulator r_∞ induces a long exact seq.

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tensor w/ $\mathbb{R}$ over $\mathbb{Q}$

$$0 \to H^0_\mathcal{M}(\mathbb{Q}, M)_\mathbb{R} \to \operatorname{Ker} \alpha_M \xrightarrow{r_\infty} H^1_\mathcal{M}(\mathbb{Z}, M^\vee(1))^\vee_\mathbb{R} \to$$

$$\to H^1_\mathcal{M}(\mathbb{Z}, M)_\mathbb{R} \xrightarrow{r_\infty} \operatorname{coker} \alpha_M \to H^0_\mathcal{M}(\mathbb{Q}, M^\vee(1))^\vee_\mathbb{R} \to 0.$$

encodes the height pairing

Example: $M = h^0(F)(r)$ then $\tan_M = \begin{cases} 0 & r \leq 0 \\ M_{dR} & r \geq 1. \end{cases}$

Case 1: $r \leq 0$, then

$$0 \to H^0_\mathcal{M}(\mathbb{Q}, h^0(F)(r))_\mathbb{R} \to h^0(F)(r)^+_{B,\mathbb{R}} \to H^1_\mathcal{M}(\mathbb{Z}, h^0(F)(1-r))^\vee_\mathbb{R} \to 0$$

$\quad \| $

$\quad 0$ if $r \neq 0$.

Case 2: $r \geq 1$, then

$$0 \to H^1_\mathcal{M}(\mathbb{Z}, h^0(F)(r))_\mathbb{R} \xrightarrow{r_\infty} h^0(F)(r-1)^+_{B,\mathbb{R}} \to H^0_\mathcal{M}(\mathbb{Q}, h^0(F)(1-r))^\vee_\mathbb{R} \to 0$$

$$K_{2r-1}(\mathcal{O}_F)_\mathbb{R} \xrightarrow{r_\infty} \Big(\bigoplus_{\sigma: F \to \mathbb{C}} (2\pi i)^{r-1}\mathbb{R}\Big)^+ \qquad 0 \text{ if } r \neq 1.$$

Thm: (Borel): Conj 3 is true for Artin motives.

Example: $r = 1$, then $r_\infty : \mathcal{O}_F^\times \otimes \mathbb{Q} \to \Big(\bigoplus_{\sigma: F \to \mathbb{C}} \mathbb{R}\Big)^+$

which is really the Dirichlet regulator.

$r_p : \mathcal{O}_F^\times \otimes \mathbb{Q} \to H^1_{cont}(\mathbb{Q}, h^0(F)(r))$ coincides with the Kummer map,

I.4 TNC:

Def: Let $L_f(M) := \det_E H^0_\mathcal{M}(\mathbb{Q}, M) \otimes \det_E^\vee H^1_\mathcal{M}(\mathbb{Z}, M)$

1-dim E.v.s. $\Delta_f(M) := L_f(M) \otimes L_f(M^\vee(1)) \otimes \det_E^\vee M_B^+ \otimes \det_E \tan_M$.

is the <u>fundamental line</u>.

<u>Conj (TNC) (Beilinson-Bloch-Kato (after Fontaine-Perrin-Riou))</u>: Assume conj 1-3, then one has

$$\iota_\infty : \Delta_f(M) \otimes_{\mathbb{Q}} \mathbb{R} \cong E \otimes_E \mathbb{R} \qquad \text{(via conj 3)}$$

$$\forall p \quad \iota_p : \Delta_f(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \det_{E \otimes \mathbb{Q}_p} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], M_p)$$

Then

a) $\exists\ \zeta_M \in \Delta_f(M)$ s.t. $\iota_\infty(\zeta_M \otimes 1) = \frac{1}{L(M,0)^*}$

(this is really Beilinson's conj)

b) $\forall p \quad \iota_p(\zeta_M \otimes 1)$ generates the $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ lattice $\det_{\mathcal{O}_E \otimes \mathbb{Z}_p} R\Gamma_c(\mathbb{Z}[\tfrac{1}{fp}], T_p)$.

This will be explained further next time.

The Tamagawa Number Conjecture and Iwasawa Main conjectures II:

Conj: (TNC): Assume 1) - 3), then one has

$$\iota_\infty : \Delta_f(M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} E \otimes_{\mathbb{Q}} \mathbb{Q} \qquad \text{(via conj 3)}$$

$$\forall p \quad \iota_p : \Delta_f(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \det_{E\otimes\mathbb{Q}_p}(R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], M_p))$$

Then a) $\exists \ \zeta_M \in \Delta_f(M)$ s.t. $\iota_\infty(\zeta_M \otimes 1) = \frac{1}{L(M,0)^*}$

b) $\forall p \quad \iota_p(\zeta_M \otimes 1)$ generates the $\mathcal{O}_p := \mathcal{O}_E \otimes \mathbb{Z}_p$ - lattice

$$\det_{\mathcal{O}_E\otimes\mathbb{Z}_p} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], T_p) \subset \det_{E\otimes\mathbb{Q}_p} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], M_p).$$

Def: Let f be the product of primes s.t. M_p is unramified outside of f, $T_p \subset M_p$ is a $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-stable ~~lattice~~ $\mathcal{O}_E \otimes \mathbb{Z}_p$ - lattice in M_p. Define

$$R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], T_p) := \text{mapping fibre}\left(R\Gamma(\mathbb{Z}[\tfrac{1}{pf}], T_p) \longrightarrow \bigoplus_{l \mid fp\infty} R\Gamma(\mathbb{Q}_l, T_p)\right).$$

Remark: The conjecture is indep. of T_p because $\chi(R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], \text{finite})) = 1$.

Theorem: TNC holds for $h^0(F)$, $h^0(F)(1)$ $\forall p \neq 2$ (Class number formula) and for $F/\mathbb{Q}$ any number field.

Theorem (Burns - Greither, Huber-Kings). TNC holds for $V(\chi)(r)$ $\forall p \neq 2$.

Cor: (cohomological Lichtenbaum conj.) $F/\mathbb{Q}$ an abelian number field, $r \geq 2$,

$$R_r(F) := \mathrm{vol}\left(h^0(F)(r-1)^+_{B,\mathbb{R}} \big/ r_\infty(K_{2r-1}(\mathcal{O}_F))\right).$$

Then

$$\frac{\zeta_F(1-r)^*}{R_r(F)} \cdot \mathbb{Z}_p \underset{\text{in } \mathbb{Q}_p}{=} \frac{\# H^2(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r))}{\# H^1(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r))_{tors}} \cdot \mathbb{Z}_p$$

an equality of lattices.

$\forall p \neq 2$.

Remark: $H^1(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(1)) = \mathcal{O}_F[\tfrac{1}{p}]^\times \otimes \mathbb{Z}_p$ (by Kummer theory)

$H^2(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(1)) = \mathrm{Pic}[\mathcal{O}_F[\tfrac{1}{p}]] \otimes \mathbb{Z}_p$

Proof (Sketch): $V = h^0(F)$.

$$\Delta_f(V(1-r)) = \det{}^\vee_{\mathbb{Q}} H^1_\mu(\mathbb{Z}, V(r)) \otimes \det{}^\vee_{\mathbb{Q}} V_B(1-r)^+ \quad (r \geq 2)$$

$$\Omega := K_{2r-1}(\mathcal{O}_F)^{\text{free}}$$

$$T_B := \bigoplus_{\sigma: F \to \mathbb{C}} \mathbb{Z} \subseteq V_B.$$

By def. $\det_{\mathbb{Z}} \Omega = R_r(F) \det_{\mathbb{Z}} T_B(r-1)^+$ in $\Delta_f(V(1-r))_{\mathbb{R}}$.

$$= R_r(F) \det{}^\vee_{\mathbb{Z}} T_B(1-r)^+$$

$$\Rightarrow \zeta_{V(1-r)} \mathbb{Z} = \frac{R_r(F)}{\zeta_F(1-r)^*} \det{}^\vee_{\mathbb{Z}} \Omega \otimes \det{}^\vee_{\mathbb{Z}} T_B(1-r)^+ \in \Delta_f(V(1-r))_{\mathbb{R}}.$$

(Prop: $\det_{\mathbb{Z}_p} R\Gamma_c(\mathbb{Z}[\tfrac{1}{p}], T_p(1-r)) = \det_{\mathbb{Z}} R\Gamma(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r)) \otimes \det{}^\vee_{\mathbb{Z}_p} T_p(1-r)^+$.)

As TNC gives

$$\frac{R_r(F)}{\zeta_F(1-r)^*} \cdot \mathbb{Z}_p = \det_{\mathbb{Z}_p} r_p(\Omega \otimes \mathbb{Z}_p) \otimes \det_{\mathbb{Z}_p} R\Gamma(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r)).$$

$$= \det{}^\vee_{\mathbb{Z}_p} \left(H^1(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r)) / r_p(\Omega \otimes \mathbb{Z}_p) \right) \otimes \det_{\mathbb{Z}_p} H^2(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r)).$$

$$= \det{}^\vee_{\mathbb{Z}_p} H^1(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r))_{\text{tors}} \otimes \det_{\mathbb{Z}_p} H^2(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(r)).$$

$$= \frac{\# H^1_{\text{tors}}}{\# H^2} \qquad \square$$

II Iwasawa Main Conjecture:

II.1: Determinants:

A any ring (with 1), $\exists$ category of virtual objects $V(A)$ and a functor

$$\det_A : \{\text{perfect complexes of } A\text{-modules}\} \longrightarrow V(A).$$
$\{$ isoms

Properties: ① $V(A)$ has comm. and assoc. $\otimes$, $\mathbb{1}_A$ unit, and M has inverse $M^\vee$.

② $V(A)$ groupoid

③ $\det_A$ is mult. on short exact seqs.

④ iso-classes in $V(A)$ are $K_0(A)$. , $\mathrm{Aut}(\mathbb{1}_A) = K_1(A) = GL_\infty(A)/E(A)$
elementary matrices.

⑤ $A \to B$ ring homom $\rightsquigarrow \otimes_A B : V(A) \longrightarrow V(B)$

⑥ X perfect complex w/ $H^i(X)$ fin-gen. proj. then

$$\det_A X = \bigotimes_i \det_A^{(-1)^i} H^i(X).$$

⑦ If A is a comm. ring and either a local ring, semi-simple, or the ring of integers of a number field, then

$$V(A) = \{(\mathcal{L}, r) \mid \mathcal{L} \text{ invert. } A\text{-module, } r: \operatorname{Spec} A \to \mathbb{Z} \text{ locally const}\}.$$

$$\det_A(M) = (\Lambda_A^{\max} M, \mathrm{rk}_A M) \quad \text{for } M \text{ f.g. proj.}$$

$$(\mathcal{L}, r) \otimes (\mathcal{L}', r') = (\mathcal{L} \otimes \mathcal{L}', r + r').$$

Fact: If A is comm. Noeth. reg., $Q(A)$ total quotient ring and f.g. M is A-torsion (i.e., $M \otimes_A Q(A) = 0$), then

$$\det_A(M) \subset (\det_A M) \otimes_A Q(A) = \det_{Q(A)}(M \otimes_A Q(A)) = Q(A).$$

determined by $(\det_A M)_{\mathfrak{p}} = \mathfrak{p}^{-\mathrm{length}_{A_\mathfrak{p}} M_\mathfrak{p}} A_\mathfrak{p} \quad \forall\, \mathrm{ht}(\mathfrak{p}) = 1$.

Example: ① $A = \mathcal{O}_E$, $M = \mathcal{O}_E / a\mathcal{O}_E$ $\quad a \in \mathcal{O}_E \quad a \neq 0$,

$$\det_{\mathcal{O}_E} M = a^{-1}\mathcal{O}_E \subset E.$$

② $A = \Lambda \cong \mathbb{Z}_p[[T]]$ (Iwasawa algebra)

M Λ-torsion, then $\det_\Lambda^\vee M = \mathrm{char}_\Lambda M$. characteristic ideal of Iwasawa theory.

Def: An isom. $\delta: \mathbb{1}_A \longrightarrow \det_A X$ is called a generator of $\det_A X$.

Notation: $\delta \in \det_A X$.

II Iwasawa Algebra and modules

$F/\mathbb{Q}$ p-adic Lie extension (i.e., $Gal(F/\mathbb{Q})$ is a p-adic Lie group, ex: $\mathbb{Z}_p^\times$, $\mathbb{Z}_p$, $GL_2(\mathbb{Z}_p)$, ...)

Write $\mathcal{O}_p := \mathcal{O}_E \otimes \mathbb{Z}_p$.

Def: $\Lambda(G) = \varprojlim_{\substack{F \supset F_n \supset \mathbb{Q} \\ \text{fin.}}} \mathcal{O}_p[Gal(F_n/\mathbb{Q})]$. This is the Iwasawa alg.

Example: ① $F = \mathbb{Q}_\infty/\mathbb{Q}$, abelian $\mathbb{Z}_p$-ext. $G = \mathbb{Z}_p$.

$\Lambda(G) \cong \mathcal{O}_p[[T]]$ non-can.

② $\mathcal{E}/\mathbb{Q}$ ell. curve non. CM, $F = \mathbb{Q}(\mathcal{E}[p^\infty])$, then

$Gal(F/\mathbb{Q}) \subset Aut(T_p\mathcal{E}) = GL_2(\mathbb{Z}_p)$.

Define: $T_p \subset M_p$ an $\mathcal{O}_p$-lattice, define $\mathbb{T}_p = T_p \otimes_{\mathcal{O}_p} \Lambda(G)$ the deformation of T_p.

Then $R\Gamma(\mathbb{Z}[\frac{1}{pt}], \mathbb{T}_p)$, $R\Gamma_c(\mathbb{Z}[\frac{1}{pt}], \mathbb{T}_p)$, $R\Gamma(\mathbb{Q}_p, \mathbb{T}_p)$ are all $\Lambda(G)$-modules.

Example: $F = \mathbb{Q}_\infty = \bigcup_{n \geq 1} \mathbb{Q}_n$, $G = Gal(F/\mathbb{Q}) = \mathbb{Z}_p$, $E = \mathbb{Q}$.

$$H^1(\mathbb{Z}[\tfrac{1}{p}], \Lambda(G) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)) \cong \varprojlim_n \mathcal{O}_{F_n}[\tfrac{1}{p}]^\times \otimes \mathbb{Z}_p.$$

$$H^2(\mathbb{Z}[\tfrac{1}{p}], \Lambda(G) \otimes \mathbb{Z}_p(1)) \cong \varprojlim_n (\mathrm{Pic}\, \mathcal{O}_{F_n}[\tfrac{1}{p}]) \otimes \mathbb{Z}_p.$$

Closely related to the classical Iwasawa modules.

II.3 Iwasawa Main conjecture: (without p-adic L-fctn).

Fix $F/\mathbb{Q}$ p-adic Lie, $G = Gal(F/\mathbb{Q})$, $T_p \subset M_p$. $\mathbb{T}_p = \Lambda(G) \otimes T_p$,
$\mathcal{O}_p = \mathcal{O}_E \otimes \mathbb{Z}_p$

Lemma: Let $M(\rho)$ be a motive s.t. $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Aut(M(\rho)_p)$ factor over G, i.e.,

$$\begin{array}{ccc} \rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & Aut(M(\rho)_p) \\ & \searrow \quad G \quad \nearrow_{\rho} & \end{array}$$

$T(\rho)_p \subset M(\rho)_p$ $\mathcal{O}_p$-lattice $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-stable and choose a basis $T(\rho)_p = \mathcal{O}_p^m$.

[Example: $G = \mathbb{Z}_p^\times$, $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathcal{O}_p^\times$ $M(\rho) = V(\chi)$]

Then

$$\Lambda(G) \otimes_{\mathcal{O}_p} T(\rho)_p \cong \Lambda(G)^n \text{ as } Gal(\overline{\mathbb{Q}}/\mathbb{Q})\text{-modules.}$$

$$g \otimes t \longmapsto g \otimes g^{-1} \cdot t.$$

Def: $\Lambda(G) \xrightarrow{\text{aug.}} \mathcal{O}_p$ induces $ev_\rho : \mathbb{T}_p \otimes_{\mathcal{O}_p} T(\rho)_p = \Lambda(G) \otimes_{\mathcal{O}_p} T_p \otimes_{\mathcal{O}_p} T(\rho)_p \to T_p \otimes_{\mathcal{O}_p} T(\rho)_p$

$$\text{Lemma} \to \; \cong \; \mathbb{T}_p^n$$

Observe: $\det_{\Lambda(G)} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], \mathbb{T}_p^m) = \bigotimes_{i=1}^m \det_{\Lambda(G)} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], \mathbb{T}_p)$

$$\delta^{\otimes m} \longleftarrow\!\!\!\shortmid \delta$$

ev_ρ induces: $ev_\rho : \det_{\Lambda(G)} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], \mathbb{T}_p^m) \otimes_{\Lambda(G)} \mathcal{O}_p \xrightarrow{\sim} \det_{\mathcal{O}_p} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], T_p \otimes T(\rho)_p)$.

Conj. (Iwasawa MC) (Huber-Kings, Kato-Fukaya): $\exists\ \zeta_{\mathbb{T}_p} \in \det_{\Lambda(G)} R\Gamma_c(\mathbb{Z}[\tfrac{1}{pf}], \mathbb{T}_p)$.
s.t. $\forall\ M(\rho)$ as above $ev_\rho(\zeta_{\mathbb{T}_p}^{\otimes m}) = \zeta_{M \otimes M(\rho)} \otimes 1)$

II.4 Relation to classical Iwasawa MC:

$F = \mathbb{Q}(\zeta_{p^\infty})$ $\quad G = \Delta \times \Gamma$ $\quad \Delta = \mathbb{Z}/_{(p-1)\mathbb{Z}}$ $\quad \Gamma = \mathbb{Z}_p$

$\Lambda(G) = \mathbb{Z}_p[\Delta][[T]]$

ρ could be $\begin{cases} \chi : (\mathbb{Z}/_{p^n\mathbb{Z}})^\times \longrightarrow \mathbb{C}^\times \\ \text{power of cyclotomic char } \varepsilon : G \longrightarrow \mathbb{Z}_p^\times \end{cases}$

$\Rightarrow \quad T(\rho)_p = \begin{cases} T(\chi)_p \\ \mathbb{Z}_p(n) \end{cases}$

<u>Lemma:</u> $\det_{\Lambda(G)} R\Gamma_c(\mathbb{Z}[\tfrac{1}{p}], \Lambda(G) \otimes \mathbb{Z}_p(1)) = \det_{\Lambda(G)} R\Gamma(\mathbb{Z}[\tfrac{1}{p}], \Lambda(G) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p) \otimes \det^\vee_{\Lambda(G)} T_p^+.$

$r_p(\zeta_{h^0(\mathbb{Q}(\zeta_{p^n}))(1)})$ give the cyclotomic units in $H^1(\mathbb{Z}[\tfrac{1}{p}], \Lambda(G) \otimes \mathbb{Z}_p(1))$.

MC for $\Lambda(G)$ and $M = \mathbb{Q}_p(1)$ $\Leftrightarrow$ $\det^\vee_{\Lambda(G)} \left(H^1(\mathbb{Z}[\tfrac{1}{p}], \Lambda(G) \otimes \mathbb{Z}_p(1)) / c\Lambda(G) \right)$

$= \det^\vee_{\Lambda(G)} H^2(\mathbb{Z}[\tfrac{1}{p}], \Lambda(G) \otimes \mathbb{Z}_p(1)).$