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PS1

Arithmetic of Picard modular surfaces modulo an inert prime

Joint w/ E. Goren

- related to work of E. Eischen.
- must generalize to $U(n, n-1)$.
- Goal: p-adic modular forms, diff operators, p-adic L-fctns, Gal. reps.

E quad. imag. field, $V = E^3$

$(u, v) = {}^t\bar{u} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} v$ w/ signature $(2,1)$

$\underline{G} = GU(V, (,)) / \mathbb{Q}$.

$G_\infty \curvearrowright \mathfrak{X}$ = unit ball in $\mathbb{C}^2$

$\ni x_0$

$K_\infty = \mathrm{Stab}(x_0)$

$K_f \subseteq \underline{G}(\mathbb{A}_f)$ principal cong. subgrp. of $N \geq 3$.

$K = K_\infty K_f \subseteq \underline{G}(\mathbb{A})$

Shimura variety Picard surface $S_{E/E}$.

$S_E(\mathbb{C}) = \underline{G}(\mathbb{Q}) \backslash \underline{G}(\mathbb{A}) / K$

This has an integral model $/R_0$, $R_0 = \mathcal{O}_E[\frac{1}{2ND_E}]$.

$\bar{S}/R_0$ smooth arithmetic compactification. (Larsen, Bellaiche, Lan)

∂S = elliptic curves with CM by $\mathcal{O}_E$.

<u>Moduli problem</u> (PEL)

S classifies $\underline{A} = (A, \lambda, \iota, \alpha) / R / R_0$

A = abelian 3-fold

$\lambda : A \to A^t$ principal polarization

$\iota : \mathcal{O}_E \hookrightarrow \mathrm{End}(A/R)$

$\mathrm{Rosati}_\lambda(a) = \bar{a}$

$\mathrm{Lie}(A/R)$ type $(2,1)$

α level N-structure.

There is a universal abelian variety

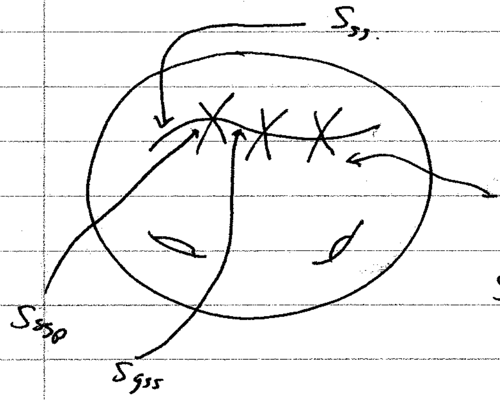
$$\begin{array}{c}\mathcal{A}\\ \downarrow\\ S.\end{array}$$

p inert in E, good.

$k_0 = R_0/pR_0$

$k = \bar{k}_0$.

$S/_k = S$.



$S_\mu = \mu$-ordinary

S_{ss} = supersingular

$= S_{gss} \cup S_{ssp}$

$S_{ssp} = \mathrm{sing}(S_{ss})$

$$x \in S_\mu \Leftrightarrow A_x[p^\infty] \cong \mathcal{O}_E \otimes \mu_{p^\infty} \oplus \mathcal{G} \oplus \mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

p-div grp of a ss. ell. curve $/k$

$$x \in S_{ss} \Leftrightarrow A_x[p^\infty] \sim \mathcal{G}^3$$

$$x \in S_{ssp} \Leftrightarrow A_x[p^\infty] \cong \mathcal{G}^3$$

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Vollard: $N \geq N_0(p)$

- irred cpts of S_{ss} smth $\cong \mathcal{C} = \{ X^{p+1} + y^{p+1} + z^{p+1} = 0 \}$
- intersect transversally
- $p+1$ branches through each $x \in S_{ssp}$
- p^3+1 ssp points on each component.

Thm: (dS - Goren: JNT): # irred. components $(S_{ss}) = \# c_2(\bar{S})/3$

Holzapfel: $= * [\Gamma(1) : \Gamma(N)] \; L(3, (\frac{D_E}{\cdot}))$.

Hasse - invariants: $\Sigma, \bar{\Sigma} : \mathcal{O}_E \to \kappa_0 \hookrightarrow k$.

$$\Omega^1_{A/\bar{S}} = \omega = \underset{2}{\omega(\Sigma)} \oplus \underset{1}{\omega(\bar{\Sigma})} = \mathcal{P} \oplus \mathcal{L}$$

$\det \mathcal{P} \cong \mathcal{L}$

$\mathrm{Ver} : A^{(p)} \to A$

$$\begin{tikzcd} A^{(p)} \arrow[r] \arrow[d] & A \arrow[d] \\ S \arrow[r, "\Phi"'] & S \end{tikzcd}$$

$V : \omega \to \omega^{(p)}$

$V_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}^{(p)}$ $\qquad h = V_{\mathcal{P}}^{(p)} \cdot V_{\mathcal{L}}$.

$V_{\mathcal{L}} : \mathcal{L} \to \mathcal{P}^{(p)}$ $\qquad$ mod form at p^2-1.

$\in H^0(\bar{S}, \mathcal{L}^{p^2-1}) = M_{p^2-1}(\bar{S}, k)$

Prop: (i) $V_{\mathcal{P}}, V_{\mathcal{L}}$ are both rank 1 outside S_{ssp}.

(ii) $\mathrm{div}(h) = S_{ss}$ (reduced)

Goldring - Nicole (2013) (PEL)

Koskivirta - Wedhorn (2014) (Hodge)

<u>Kodaira - Spencer map</u>

$KS: \mathcal{P} \otimes \mathcal{L} \simeq \Omega_S^1$

$\mathcal{P}_0 = \ker V_{\mathcal{P}}$ line bundle out S_{ssp}.

$\pitchfork$ at ∂S (transversal)

$/\!/$ at S_{gss} (tangential)

Secondary Hasse inv defined only on S_{ss}, vanishes only at S_{ssp} (general Boxer, Koskivirta ..) on S_{gss}:

$$V_{\mathcal{L}} \simeq \mathcal{L} \simeq \mathcal{P}_0^{(p)}$$

$$V_{\mathcal{P}}^{(p)} : (\mathcal{P}/\mathcal{P}_0)^{(p)} \simeq \mathcal{L}^{(p^2)}$$

$$h_{(2)} = V_{\mathcal{P}}^{(p)} \otimes V_{\mathcal{L}}^{-1} : (\det \mathcal{P})^{(p)} = \mathcal{L}^{(p)} \simeq \mathcal{L}^{p^2+1}$$

$$h_{(2)}' \in H^0(S_{gss}, \mathcal{L}^{p^2-p+1})$$

$$h_{(2)} = (h_{(2)}')^{p+1} \in H^0(S_{ss}, \mathcal{L}^{p^3+1})$$

$\operatorname{div}(h_{(2)})$ supp on S_{ssp}.

<u>Igusa surface of level p</u>

$$\begin{array}{ccc} Ig_p & \subseteq & \overline{Ig}_p \\ \tau \downarrow & & \downarrow \\ S_p & \subseteq & \overline{S}_p \end{array}$$

étale Galois $\Delta = Gal(\tau) = \kappa_0^*$

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classifies
$$\epsilon : \mathcal{O}_E \otimes \mu_p \hookrightarrow A[p]$$

Compactifies over S_{ss} $\quad \bar{I}_g \xrightarrow{\tau} \bar{S}$.

- normal surface, sing / S_{ssp}
- tot. ram. / S_{ss}
- relatively irred.
- key pt. $\tau^* h = a^{p^2-1}$

a is canonical section trivializing $\tau^* \mathcal{L}$.

<u>The theta operator</u>

$$\begin{array}{ccccc}
H^0(Ig_\mu, \mathcal{O})^{(k)} & \xrightarrow{d} & H^0(Ig_\mu, \Omega^1)^{(k)} & \underset{KS}{\xleftarrow{\sim}} & H^0(Ig_\mu, P\otimes\mathcal{L})^{(k)} \\
{\scriptstyle f \mapsto \tau^* f / a^k} \uparrow & & & & \downarrow V_P \otimes 1 \\
H^0(S, \mathcal{L}^k) & & & & H^0(Ig_\mu, \mathcal{L}^{p+1}) \\
\| & & & & \downarrow a^k \\
M_k(N; k) & \xrightarrow{\Theta} & M_{k+p+1}(N; k) & \hookrightarrow & H^0(S_\mu, \mathcal{L}^{k+p+1})
\end{array}$$

Thm: (i) Θ is well-defined (extends holomorphically across S_{ss})

(ii) Θ is a derivation, image $\subseteq$ cusp forms.

(iii) Θ is compatible with the Serre-Katz Θ operator on embedded modular curves

(iv) On FJ expansions at cusps $\Theta =$ "$q \frac{d}{dq}$"

$$f|_C = \sum_{m=0}^{\infty} \Theta_m(u) q^m.$$

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B. Moonen generalized Serre-Tate theory to all Shimura varieties on the μ-ordinary locus.

- cascades
- :

Moonen
$\Longrightarrow$ in our S, $x \in S_\mu(k)$ formal nbd. of x is canonically a $\hat{\mathbb{G}}_m$-torsor over $\hat{\mathcal{D}}$.

(v) Ⓗ "at x" is "the translation invariant derivation in the $\hat{\mathbb{G}}_m$-direction."