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On Galois representations of mod p Hilbert eigenforms of weight 1

joint w/ M. Dimitrov

$$\left\{\begin{array}{l}\text{Hecke eigenforms} \\ \text{over } \overline{\mathbb{F}}_p, \ \sum a_n q^n \in \overline{\mathbb{F}}_p[[q]]\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p) \text{ odd} \\ \text{semisimple}\end{array}\right\}$$

min level $\longleftrightarrow$ conductor

min wt. $\longleftrightarrow$ "Serre wt" of ρ

wt 1 $\longleftrightarrow$ ρ unramified at p

What about Hilbert modular forms?

F totally real field. Consider geometrically defined Hilbert modular forms over $\overline{\mathbb{F}}_p$, only work in parallel weight.

Think of HMF as adelic q-expansions:

$$f = \sum_{\mathfrak{b} \triangleleft \mathcal{O}_F} a_{\mathfrak{b}} q^{\mathfrak{b}} \in \overline{\mathbb{F}}_q[[q^*]]$$

(nice way to work for Hecke theory.)

<u>Thm (Dimitrov - W.):</u> Let $f \in M_1(\mathfrak{n}, \psi, \overline{\mathbb{F}}_p)$ weight 1 Hilbert e.f. of level $\mathfrak{n}$, $(p, \mathfrak{n}) = 1$, and nebentypus ψ. Then the Galois representation

$$\rho_f : G_F \longrightarrow GL_2(\overline{\mathbb{F}}_p)$$

is unramified outside $\mathfrak{n}$. In particular, it is unramified above p. For all prime ideals $\mathfrak{q} \nmid \mathfrak{n}$ ($\mathfrak{q} = \mathfrak{p} | p$ allowed) have char poly. $(\rho_f(\mathrm{Frob}_{\mathfrak{q}})) = X^2 - a_{\mathfrak{q}} X + \psi(\mathfrak{q})$.

<u>Remark:</u> Emerton, Reduzzi, Xiao prove Thm if ρ_f is p-distinguished and p inert in F.

<u>Thm</u> (Sasaki, Pilloni, Kassaei, Gee, Sasaki, ...) Let $\rho : G_F \to GL_2(\overline{\mathbb{F}}_p)$ modular, irred, p-distinguished, ..., Then ← unram above p
there exists $g \in M_1(\mathfrak{n}, \psi; \overline{\mathbb{F}}_p)$ eigenform s.t.

$$\rho \simeq \rho_g.$$

How to prove theorem?

Ingredients:
- Hasse invariant
- $\exists$ lifts in high enough weights
- Local description above p of ordinary modular Galois representations.
- Hecke operator $T_{\mathfrak p}$ for $\mathfrak p \mid p$ on geometric Hilbert modular forms

(we define them and compute the action on adelic q-exp; relies on a proposition in Emerton - Reduzzi - Xiao. Pilloni has an alternative construction.)

$$\begin{tikzcd} M_1(\mathfrak{n}, \psi, \overline{\mathbb{F}}_p) \arrow[r, "V_{\mathfrak p}"'] & M_p(\mathfrak{n}, \psi, \overline{\mathbb{F}}_p)^{ord} \arrow[r, "h^{k_0-1}"] & M_k(\mathfrak{n}, \psi, \overline{\mathbb{F}}_p)^{ord} \\ & & M_k(\mathfrak{n}, \psi, \overline{\mathbb{F}}_p)^{ord} \arrow[u] \end{tikzcd}$$

Hasse invariant $h \in M_{p-1}(1, \overline{\mathbb{F}}_p)$ with q-expansion 1.

$T_{\mathfrak p}$ acts: $a_{\mathfrak b}(T_{\mathfrak p} f) = a_{\mathfrak{b}\mathfrak{p}}(f) + N(\mathfrak p)^{k-1} \psi(\mathfrak p)\, a_{\mathfrak{b}/\mathfrak{p}}(f)$.

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<u>Prop:</u> Let $\mathfrak{P} | p$ be a square free ideal. There is a map

$$V_{\mathfrak{P}} : M_1(\mathfrak{n}, \psi; \overline{\mathbb{F}}_p) \longrightarrow M_p(\mathfrak{n}, \psi, \overline{\mathbb{F}}_p)$$

commuting with all $T_{\mathfrak{q}}$ for $\mathfrak{q} \nmid p$ st.

$$a_{\mathfrak{b}}(V_{\mathfrak{P}} f) = a_{\mathfrak{b}/\mathfrak{P}}$$

<u>Idea of proof:</u> $\mathfrak{p} | p$ prime.

$$a_{\mathfrak{b}}(h T_{\mathfrak{p}}^{(1)} f) = a_{\mathfrak{b}\mathfrak{p}}(f) + \psi(\mathfrak{p}) a_{\mathfrak{b}/\mathfrak{p}}$$

$$a_{\mathfrak{b}}(T_{\mathfrak{p}}^{(p)} h f) = a_{\mathfrak{b}\mathfrak{p}}(f).$$

$$\Rightarrow a_{\mathfrak{b}/\mathfrak{p}}(f) = a_{\mathfrak{b}}\Big(\underbrace{\psi(\mathfrak{p})^{-1} (h T_{\mathfrak{p}}^{(1)} - T_{\mathfrak{p}}^{(p)} h)}_{V_{\mathfrak{p}}} f \Big).$$

<u>General def:</u> $V_1 = h.$

$$V_{\mathfrak{P}\mathfrak{p}} = \psi(\mathfrak{p})^{-1} (V_{\mathfrak{P}} T_{\mathfrak{p}}^{(1)} - T_{\mathfrak{p}}^{(p)} V_{\mathfrak{P}}).$$

normalized e.f.

<u>Prop:</u> (a) Let $f \in M_1(\dots)^{\checkmark}$. Then the $V_{\mathfrak{P}}(f)$ for $\mathfrak{P} | p$ square-free are linearly indep.

(b) $W = \langle V_{\mathfrak{P}}(f) : \mathfrak{P} | p \text{ square free} \rangle$

W is $T_{\mathfrak{p}}$-stable for all $\mathfrak{p} | p$.

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The minimal polynomial of T_p acting on W equals

$$X^2 - a_p(f) X + \psi(p).$$

. T_p is invertible so W is ordinary.

$$T_q|_W = a_q \quad \forall q | p.$$

Let $k = 1 + k_0(p-1)$ be big enough.

Let $\mathbb{T}$ be the $\mathbb{Z}_p$-algebra acting on $M_k(n, \psi; \overline{\mathbb{Z}}_p)^{\text{ord}}$ generated outside np.

Let m be the maximal ideal of $\mathbb{T}$ generated by p and $T_\ell - a_\ell(f)$ for $(\ell, np) = 1$.

$\mathbb{T}_m$ = localization of $\mathbb{T}$ at m.

$$\mathbb{T}_m \otimes \overline{\mathbb{Q}}_p \cong \prod_{g \text{ newform}} \overline{\mathbb{Q}}_p \qquad T_\ell \longmapsto (\ldots, a_\ell(g), \ldots)$$

$$T_p \in \mathbb{T}_m \otimes \overline{\mathbb{Q}}_p \quad \forall p | p.$$

Wiles on ordinary Galois reps:

$$\exists \ \rho_m : G_F \longrightarrow GL_2(\mathbb{T}_m \otimes \overline{\mathbb{Q}}_p) = GL(V) \quad \text{s.t.}$$

- unramified outside np
- $Tr(\rho_m(\mathrm{Frob}_q)) = T_q \quad \forall q \nmid np$

$$\begin{array}{ccccccccc} & & \mathbb{T}_m \otimes \overline{\mathbb{Q}}_p & & (\mathbb{T}_m \otimes \overline{\mathbb{Q}}_p)^2 & & \mathbb{T}_m \otimes \overline{\mathbb{Q}}_p & & \\ & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \\ 0 & \longrightarrow & V^+ & \longrightarrow & V & \longrightarrow & V^- & \longrightarrow & 0 \end{array}$$

unram. above p and $\mathrm{Frob}_p = T_p$ $\forall p | p$.

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$$0 \to V^+ \to V \to V^- \to 0$$

$$\begin{array}{ccccccccc} & & \cup\!\mid \text{lattice} & & \cup\!\mid \text{lattice} & & \cup\!\mid & & \\ 0 & \to & \mathcal{L}^+ & \to & \mathcal{L} & \to & \mathcal{L}^- & \to & 0 \\ & & & & \wr\!\mid\!\mid & & & & \\ & & & & \mathbb{T}_m^2 & & & & \end{array}$$

$\Rightarrow$

$$\mathcal{L}^+/m \longrightarrow \mathcal{L}/m \longrightarrow \mathcal{L}^-/m \longrightarrow 0$$

$$\mathcal{L}/m = \rho_f. \quad 2\text{-dim over } \mathbb{T}/m = \mathbb{F}.$$

if $\dim \mathcal{L}^-/m = 0 \overset{\text{Naka.}}{\Rightarrow} \mathcal{L}^- = 0 \quad \#$

if $\dim \mathcal{L}^-/m = 1 \overset{\text{Naka}}{\Rightarrow} \mathbb{T}_m \overset{\sim}{\twoheadrightarrow} \mathcal{L}^-$

$T_p = \mathrm{Frob}_p$ on $\mathcal{L}^- \Rightarrow T_p \in \mathbb{T}_m$

$\Rightarrow T_p|_{h^{k-1}W} \in \mathbb{F}.$

But min. poly. has degree 2, over $\mathbb{F}$. #.

$\Rightarrow \dim \mathcal{L}^-/m = 2. \Rightarrow \underset{\text{unram.}}{\mathcal{L}^-/m} \cong \rho_f$

$\blacksquare$