Euler systems and deformations of Galois representations

M motive over $\mathbb{Q}$
M_p p-adic realization, M_p $\mathcal{O}$ lattice, $\mathcal{O}/\mathbb{Z}_p$
$L(s,M)$ L-fctn.
d integer
Euler system of rank d collection of classes

$$Z_m \in \Lambda^d H^1(\mathbb{Q}(\mu_m)^+, M_p)$$

Norm relations
ℓ unramified prime

$$\mathrm{Cores}_{\mathbb{Q}(\mu_m)^+}^{\mathbb{Q}(\mu_{m\ell})^+} Z_{m\ell} = \begin{cases} Z_m & \text{if } \ell \mid m \\ P_\ell(\mathrm{Frob}_\ell) Z_m & \text{if } \ell \nmid m \end{cases}$$

$$P_\ell(X) = \det(1 - \mathrm{Frob}_\ell X \mid_{M_p^*(1)})$$

Adjoint modular Galois representation

F tot real field of degree d.

f Hilbert modular form of wt k (parallel)
p splits in F and f is ordinary at $v \mid p$.

Prop: Let $\rho_f : G_F \to GL_2(\mathcal{O})$. Assume that $\bar\rho_f$ abs. irred.
Then there exists a canonical element

$$Z_1 \in \Lambda^d H^1(F, ad(\rho_f))$$

Sketch: Hida theory
$\mathbb{T}$ local component of the universal ordinary Hecke algebra for F attached to f.

$\Lambda_d = \mathcal{O}[\![T_v : v | p]\!]$ (T_v variables, not Hecke ops)

$T_v = 0$ corresponds to wt $(k, \dots, k)$, $v | p$.

$\rho_{\mathbb{T}} : G_F \longrightarrow GL_2(\mathbb{T})$ $\qquad \mathbb{T}/(T_v : v|p) \cong \mathbb{T}_k^{\mathrm{ord}}$

$\lambda_f : \mathbb{T} \to \mathcal{O}$ Hecke char. corresponding to f.

$\sigma \in G_F$: $\tilde{C}(\sigma) := \rho_f(\sigma)\, \rho_{\mathbb{T}}(\sigma)^{-1} (\det \rho_{\mathbb{T}} \det \rho_f^{-1})^{1/2} - 1_2$
$\in I \cdot M_2(\mathbb{T})$

$I = \ker(\lambda_f)$.

$C(\sigma) = \tilde{C}(\sigma) \pmod{I^2}$.

$\sigma \longmapsto C(\sigma)$ is a 1-cocycle taking values in $I/I^2 \otimes \mathrm{ad}\, \rho_f$.

$I/I^2 \cong \Omega_{\mathbb{T}/\mathcal{O}} \otimes_{\lambda_f} \mathcal{O}$

$$\begin{array}{ccccccccc}
 & \bigoplus_{v|p} \mathcal{O}\, dT_v & & L' & & \Omega_{\mathbb{T}_k/\mathcal{O}} \otimes \mathcal{O} & & & \\
 & \| & & \| & & \| & & & \\
0 \to & \Omega_{\Lambda_d/\mathcal{O}} \otimes \mathcal{O} & \longrightarrow & \Omega_{\mathbb{T}/\mathcal{O}} \otimes \mathcal{O} & \longrightarrow & \Omega_{\mathbb{T}/\Lambda_d} \otimes_{\lambda_f} \mathcal{O} & \longrightarrow 0 & &
\end{array}$$

C_f = congruence number attached to f.

$$= \frac{L(1, \mathrm{ad}(\rho_f))}{\Omega_f^+ \Omega_f^-}.$$

L' is NOT a canonical lattice.

$\forall v | p$
$c_v(\sigma) = \left(C(\sigma), \frac{d}{dT_v} \right) \in \mathrm{ad}(\rho_f)$.

$\bigwedge_v c_v \in \Lambda^d L'$

$$Z_1 = \# \left(\Omega^1_{\Pi_F/\mathcal{O}} \otimes \mathcal{O} \right) \bigwedge_{v|p} c_v(\sigma)$$

$$\in \Lambda^d H^1(F, ad(\rho_f)). \blacksquare$$

(defined up to a p-adic unit)

<u>We now restrict to $F = \mathbb{Q}$.</u>

$m > 1$ $\quad \mathbb{Q}(\mu_m)^+ \quad \hat{f}$ base change of f to $\mathbb{Q}(\mu_m)^+$

$\Rightarrow \quad c_m \in H^1(\mathbb{Q}(\mu_m)^+, ad(\rho_f)) \otimes \Omega_{\Pi_{\mathbb{Q}(\mu_m)^+}/\mathcal{O}} \otimes \mathcal{O}.$

module over $A_m = \mathcal{O}[\Delta_m]$

$$\Delta_m = Gal(\mathbb{Q}(\mu_m)^+/\mathbb{Q}).$$

$$= (\mathbb{Z}/m\mathbb{Z})^\times / \langle \pm 1 \rangle$$

There exists an element

$$\Theta_m := L(ad(\rho_f) \otimes \langle , \rangle_m, 1) \in A_m$$

s.t. $\forall\ \psi : \Delta_m \longrightarrow \bar{\mathbb{Q}}_p^\times$ we have

$$\psi\left(L(ad(\rho_f) \otimes \langle , \rangle_m, 1) \right) = \frac{L^{*}(1, ad(\rho_f) \otimes \psi)}{\Omega_f^+ \Omega_f^-}.$$

<u>Conj:</u> Θ_m annihilates $\Omega_{T\mathbb{Q}(\mu_m)^+/\mathcal{O}} \otimes \mathcal{O}$.

Urban

8-17-16

pg 4

<u>Prop:</u> If the conjecture holds, $Z_m = \Theta_m c_m \in H^1(\mathbb{Q}(\mu_m)^+, ad(\rho_f))$

is an Euler system.

<u>Another Example:</u>

Congruences between critical E.S. and cuspforms.

$$\psi : \Delta_m \longrightarrow \overline{\mathbb{Q}}_p^\times$$

$E^{crit}_{k,\psi}$ level mp and $E^{crit}_{k,\psi} | U_p = p^{k-1} E^{crit}_{k,\psi}$.

Via Coleman get a family and so using Ribet

$$\Rightarrow \begin{pmatrix} \varepsilon^{k-1} & \overset{\neq 0}{*} \\ 0 & \psi \end{pmatrix}$$

This gives an extension, so an element of $H^1(\mathbb{Q}, L(k-1)(\psi^{-1}))$

p-adic field containing values of ψ

$= H^1(\mathbb{Q}(\mu_m)^+, L(k-1))^{\psi^{-1}} \ni c_{\psi,m} \neq 0.$

Idea: There should be a way to patch the

$c_{\psi,m}$, $\psi \in \Delta_m^*$ s.t. $c_m \in H^1(\mathbb{Q}(\mu_m)^+, \mathbb{Z}_p(k-1))$.

$\mathcal{C}_m$ eigencurve of level m.

Urban
8-18-16
p5 5

$$\Theta(\mathcal{C}_m) \xrightarrow{\lambda_{Eis,m}} A_m = \mathbb{Z}_p[\Delta_m]$$

$$T_\ell \longmapsto \langle \ell \rangle + \ell^{k-1}$$

$$U_p \longmapsto p^{k-1}$$

$\mathcal{V}$ = closed disc of center $[k] \in \mathcal{X}$-weight space.
and

$$\begin{array}{ccc} \mathcal{C}_{m,\mathcal{V}} & \supset & \mathcal{C}_{m,\mathcal{V}} \ (\text{slope} = k-1) \\ \downarrow & & \downarrow \\ \mathcal{V} & & \mathcal{V} \end{array}$$

may not be finite but is flat.

We shrink $\mathcal{V}$ so that the map

$$\begin{array}{c} \mathcal{C}_{m,\mathcal{V}} \\ \downarrow \\ \mathcal{V} \end{array}$$

is flat and finite on the level of the affine formal model.

needed for integrality.

We have a pseudo-representation

$$T : G_{\mathbb{Q}} \longrightarrow \mathcal{O}(\mathcal{C}_{m,\mathcal{V}}) = \mathbb{T}_{m,\mathcal{V}}$$

$$T = (a(\sigma), d(\sigma), x(\sigma,\tau))$$

$$G_{\mathbb{Q}} \longrightarrow \mathcal{O}(\mathcal{C}_{m,\mathcal{V}}) \xrightarrow[\lambda_{Eis,m}]{} A_m.$$

$\Rightarrow$

$$x(\sigma,\tau) \in I = \ker(\lambda_{Eis,m})$$

Urban

8-17-16

pg 6

? $A_m = A_N$? using A_N on board but N not defined.. yes..

$$X(\sigma,\tau) \in I \otimes A_{\mathfrak{m}} < I/I^2 = \Omega^1_{\mathbb{T}_{m,v}/A_{\mathfrak{m}}} \otimes A_{\mathfrak{m}}$$

$$X(\sigma,\tau) \in H^1(\mathbb{Q}, A_{\mathfrak{m}}(k-1)) \otimes H^1(\mathbb{Q}, A_{\mathfrak{m}}(1-k))$$
$$\otimes \Omega^1_{\mathbb{T}_{m,v}/A_{\mathfrak{m}}} \otimes A_{\mathfrak{m}}$$

$$\underbrace{\Omega^1_{\mathcal{O}(v)/\mathcal{O}} \otimes_{\mathcal{O}} A_{\mathfrak{m}}}_{\text{IS}} \to \Omega^1_{\mathbb{T}_{m,v}/A_{\mathfrak{m}}} \otimes A_{\mathfrak{m}} \to \Omega^1_{\mathbb{T}_{m,v}/A_{\mathfrak{m}} \otimes \mathcal{O}(v)} \otimes A_{\mathfrak{m}}$$

$\cong \mathcal{O} = p^v \mathcal{O}\,dT$

$A_{\mathfrak{m}}$

$\mathcal{O}(v) \supset \mathcal{O}[[T]]$

$$\Omega^{1\,(k-1)}_{\mathbb{T}_{m,k}/A_{\mathfrak{m}}} \otimes A_{\mathfrak{m}}.$$

Assuming A_m $\mathrm{Spec}(\lambda_{m_{Eis}})$ are étale

<u>Prop:</u> We can choose $\sigma, \tau \in G_{\mathbb{Q}_p}$ s.t. $X(\sigma,\tau) \neq 0$ is a 1-cocycle in σ and

$$\sigma \mapsto X(\sigma,\tau) \times L_p(k-1, \langle,\rangle_N) \in A_m$$

$\uparrow$

annihilate

$$\Omega_{\mathbb{T}_{k,m}/A_m} \otimes A_m.$$

$\Rightarrow$ we get a class

$$c_m \in H^1(\mathbb{Q}, A_m(k-1)) \otimes \mathcal{O}p^{-v}dT.$$

Conj: C_m is actually taking values in $\mathcal{O}$. and so gives an Euler system

Remark: The same construction for totally real field gives elements $Z_m \in \Lambda^d H^1(F_n, \mathbb{Z}_p(k-1))$.