Tilouine

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ps1

Congruence ideals and Selmer groups for higher symmetric powers

joint w/ Hida

① Introduction:

f eigenform wt $k = m+2$, $m \geq 0$

level $\Gamma_0(N)$. $p \nmid N$. f ordinary at p.

$$\rho_f|_{I_p} \simeq \begin{pmatrix} 1 & * \\ 0 & \chi^{-(m+1)} \end{pmatrix}.$$

$$SL_2(k') \subseteq \mathrm{Im}\,\bar{\rho}_f \subseteq GL_2(k').$$

$j \geq 1$. $a_f^j = \underset{GL_2/\mathbb{Z}_p}{Sym^{2j} \otimes \det^{-j}} \circ \rho_f$ $\qquad \bar{a}_f^j$ = red. mod p.

$a_f^j|_{I_p}$ has filtration $\underbrace{0 \subset F^+}_{>0\ \text{wt.}} \subset \underbrace{F^0}_{0\ \text{wt}} \subset \underbrace{F^-}_{<0\ \text{wts}}$

$$Sel_{(N),\mathrm{ord}}(\bar{a}_f^j) = \Big\{ c \in H^1(\mathbb{Q}, \bar{a}_f^j) \quad c_{(N)} \cdots$$
$$res_p c \in \mathrm{Im}(L_p' \to H^1(\mathbb{Q}_p, \bar{a}_f^j))$$
$$\overset{\shortparallel}{\ker}(H^1(\mathbb{Q}_p, F^-) \to H^1(I_p, F^+)) \Big\}$$

$\forall j \geq 1 \qquad p \mid L^*(a_f^j, 1) \Rightarrow p \mid \# Sel_{(N),\mathrm{ord}}(\bar{a}_f^j).$

$\Updownarrow$

$\exists g$ not colinear to f s.t. $f \equiv g \pmod{p}$

$j=1$ equivalent (Hida and Tilot)

$j > 1$ $\quad L(a_f^j, s)$ studied by Shahidi but L^* not.

We will speak in the language of congruences instead of L-functions.

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<u>Conj:</u> N square-free, ρ_f N-minimal,

$\bar{\rho}_f|_{I_\ell} \sim \begin{pmatrix} 1 & *^{\neq 0} \\ 0 & 1 \end{pmatrix}$ $\forall$ $\ell | N$.

$j = 2, 3, 4$ $C_j \Rightarrow S_j$, S_j: $p \mid \# Sel_{min, crd}(\bar{a}^j_f)$

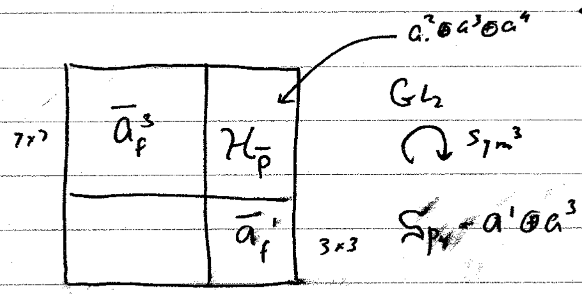
C_3: $\exists G$ cuspidal eigenform on GSp_4, level $\Gamma_{Iw}(N)$ not a Sym^3,
$Sym^3 f \equiv G \pmod{p}$.

C_2: $\exists G$ cusp. eigen on $U(4)$ not ~~[illegible]~~ coming from GSp_4 s.t. $Sym^3 f \equiv G \pmod{p}$

C_4: $\exists G \cdots$ on $U(5)$ not ~~[illegible]~~ coming from GSp_4 s.t. $Sym^4 f \equiv G \pmod{p}$.

<u>Prop:</u> $C_3 \Rightarrow S_3$ or S_2 or S_4.

← gives something in Selmer group but you don't know which one.



<u>② The theorems, in vague form:</u>

$j = 2, 3, 4$ (because only 50 minutes for talk)

N sq. free, ~~ρ~~ ρ N-minimal, $p \nmid N$.

h, Hida-Hecke algebra.

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$$\begin{tikzcd} h_1 \arrow[r, "\mu"] \arrow[dr, "\text{fin. tor. free}"'] & A_1 \ \leftarrow \text{domain} \\ & \Lambda_1 = \mathbb{Z}_p[[T]] \arrow[u, dash, "\text{fin tor. free}"'] \end{tikzcd}$$

$\tilde{A}_1$ = normal, [flat over Λ_1]

$$\tilde{h}_1 = h_1 \otimes_{\Lambda_1} \tilde{A}_1 \xrightarrow{\tilde{\mu}} \tilde{A}_1$$

$\exists\ \tilde{h}_1'$ quotient of $\tilde{h}_1$ s.t.

$$\tilde{h}_1 \hookrightarrow \tilde{A}_1 \times \tilde{h}_1'$$

$\mathcal{L}^1_\mu = \tilde{h}_1 \cap (\tilde{A}_1 \times 0)$ as ideal of $\tilde{A}_1$.

congruence ideal (between μ and some other family)

$$(L_p(a^1_\mu)) = \mathcal{L}^1_\mu = (\chi_{X^1_\mu})$$

$\left[\begin{array}{c} \text{Need} \\ \mu(U_p) = \alpha \\ \alpha^2 \not\equiv 1 \ (\mathrm{mod}\ m_{A_1}) \\ (a+1) < p-1 \end{array}\right]$

↑ cyclotomic var. fixed, only wt varying

where

$$X^j_\mu = (Sel_{min}(a^j \circ \rho_\mu \otimes_{A_1} \tilde{A}_1^*))^*$$

where $*$ means Hom to $\mathbb{Q}_p/\mathbb{Z}_p$.

f.g. $\tilde{A}_1$-module.

$j = 2, 3, 4$ $\quad \mathcal{L}^j_\mu = (\chi_{X^j_\mu})$.

③ $j=3$ precise version and proof:

$\bar{\rho}_\mu$ large image and N minimal, $p \nmid N$.

h_2 : Hida-Hecke algebra of level $\Gamma_{Iw}(N)$ for $GSp(4)$

$\Big|$ fin. tor. free.

$\Lambda_2 = \mathbb{Z}_p[[T_1, T_2]]$

$GL_2 \xhookrightarrow{Sym^3} GSp_4$

$$\begin{tikzcd} h_2 \arrow[r, "\sigma_3"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & h_1 \arrow[r, "\mu"] \arrow[d] & A_1 \\ \Lambda_2 \arrow[r] & \Lambda_1 & \end{tikzcd}$$

$\langle u \rangle = 1 + p\mathbb{Z}_p$

$1+T_1 \longmapsto (1+T)^2$

$1+T_2 \longmapsto (1+T)$

comes from localizing

$T_2 \xrightarrow{\sigma_3} T_1 \xrightarrow{\mu} A_1$

$$\begin{tikzcd} \tilde{T}_2 \arrow[r, "\tilde{\sigma}_3"] \arrow[rr, bend right, "\lambda"'] & \tilde{T}_1 \arrow[r, "\mu"] & \tilde{A}_1 \end{tikzcd}$$

$\mu \quad \tilde{T}_1 \hookrightarrow \tilde{A}_1 \times \tilde{T}_1' \qquad C_\mu^1$

$\lambda \quad \tilde{T}_2 \hookrightarrow \tilde{A}_1 \times \tilde{T}_2' \qquad C_\lambda$

$\sigma_3 \quad \tilde{T}_2 \hookrightarrow \tilde{T}_1 \times \tilde{T}_2'' \qquad C_{\sigma_3} \subset \tilde{T}_1$

$C_\mu^3 = \tilde{\mu}(C_{\sigma_3})$

T_1, T_2 are complete intersections, flat over Λ_2, A_1 (follows from

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$\alpha^2 \not\equiv 1 \pmod{m_A}$

P, Itoni: $\alpha^2 \not\equiv 1 \pmod{m_{A_1}}$

$$T_1 = R_1, \quad T_2 = R_2 .$$

Tate, Hida $\Rightarrow$ $\mathcal{L}_\lambda = \mathcal{L}_\mu^1 \mathcal{L}_\mu^3$ in $\tilde{A}_1$ all are principal.

$$\mathcal{L}_\mu^3 = (\chi_{X_\mu^3}) \qquad G_{\mathbb{Q}} \circlearrowright^{\mathrm{Sym}^3 \rho_f} \mathfrak{sp}_4 = a_\mu^1 \oplus a_\mu^3 .$$

$$\mathrm{Fitt}_{\tilde{A}_0}(X_\lambda) = \mathrm{Fitt}(\Omega_{R_2/\Lambda_2} \otimes_{\Lambda_2} \tilde{A}_1) = \mathrm{Fitt}(\Omega_{T_2/\Lambda_2} \otimes \tilde{A}_2) = \mathcal{L}_\lambda$$

where

$$X_\lambda = \mathrm{Sel}_{\min}(\cancel{}\ \mathrm{Ad}_{\mathfrak{sp}_4}\, \mathrm{Sym}^3 \rho_f \otimes \tilde{A}_1^*)$$

$a+1 < p-1$

$$X_\lambda = X_\mu^1 \oplus X_\mu^3$$

$$\mathcal{L}_\lambda = \mathrm{Fitt}\ X_\lambda$$
$$\mathcal{L}_\mu^1 = \mathrm{Fitt}\ X_\mu^1$$
$$\mathcal{L}_\mu^3 = \mathrm{Fitt}\ X_\mu^3 = (\chi_{X_\mu^3}).$$

④ $j=2$ precise statement and proof:

$$\begin{array}{ccc} & GL_4 & \\ {}^{\text{Mok}}\nearrow & & \nwarrow^{\text{Cl-Zel}} \\ GSp_4 & \longrightarrow & U(4) \end{array}$$

K arb. imag. quad. field in which p and all $l \mid N$ split.

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$D_{/K}$ div. alg. over K $d \geq 6$

ram. at $\mathcal{L}_1$, $\mathcal{L}_1^c$, $\ell_1 \| N$. $(\ell_1) = \mathcal{L}_1 \mathcal{L}_1^c$.

$\exists$ * second kind on D

$U(4) = U(D, *)$ ↗ definite

↘ quasi split at all in. prime.

split at p and $\ell | N$, $\neq \ell_1$

h_3 Hida Hecke algebra for $U(4)$, level $\Gamma_{I_\infty}(N/\ell_1)$.

Geraghty's thesis

$\boxed{3(a+1) < p \text{ is needed}}$

$\Lambda_3 = \mathbb{Z}_p[[T_1, T_2, T_3]]$

$$\begin{tikzcd} h_3 \arrow[r, "\theta"] \arrow[d, dash] \arrow[rr, bend right, "\varphi"'] & h_2 \arrow[r, "\lambda"] \arrow[d, dash] & A_1 \\ \Lambda_3 & \Lambda_2 & \end{tikzcd}$$

localize diagram to get

$$\begin{tikzcd} \tilde{T}_3 \arrow[r, "\tilde{\theta}"] \arrow[rr, bend right, "\tilde{\varphi}"'] & \tilde{T}_2 \arrow[r, "\tilde{\lambda}"] & \tilde{A}_1 \end{tikzcd}$$

$\mathcal{L}_\varphi$, $\mathcal{L}_\lambda$, $\tilde{\lambda}(\mathcal{L}_{\tilde{\theta}})$

$\mathcal{L}_\lambda = \mathcal{L}_\mu^1 \mathcal{L}_\mu^3$, $\tilde{\lambda}(\mathcal{L}_{\tilde{\theta}}) = \mathcal{L}_\mu^2$

$\mathfrak{sl}_4 = \underbrace{a^1 \oplus a^3}_{\mathfrak{sl}_{\rho_4}} \oplus a^2$ $\quad \mathcal{L}_\varphi = \mathrm{Fitt}(X_\varphi)$.

$G_\infty \circlearrowright_{\mathrm{Sym}^3 \rho_\mu}$

$\mathcal{L}_\varphi = \mathcal{L}_\mu^1 \mathcal{L}_\mu^3 \mathcal{L}_\mu^2$

$X_\varphi = X_\mu^1 \oplus X_\mu^3 \oplus X_\mu^2$.

$\Rightarrow \mathcal{L}_\mu^2 = \cancel{\mathcal{X}_\mu^2} \cdot (\mathcal{X}_{X_\mu^2})$.