Main Conjectures and Special values of L-functions

$E/\mathbb{Q}$ elliptic curve, conductor N_E

$p \geq 3$.

(H)
- $E[p]$ is an irreducible representation of $G_{\mathbb{Q}}$
- $p \nmid N_E$
- $\exists\, \ell \| N_E$ s.t. $E[p]$ is ramified at ℓ.
  (always true for semi-stable)

Thm A: Assume (H).

(a) Suppose $L(1,E) \neq 0$. Then

$$\mathrm{ord}_p\left(\frac{L(1,E)}{\Omega_E}\right) = \mathrm{ord}_p\left(\#Ш(E/\mathbb{Q}) \prod c_\ell(E)\right)$$

(b) If $L(1,E)=0$, then $\#\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = \infty$.

(c) If $\mathrm{ord}_{s=1} L(1,E) = 1$, then $\mathrm{ord}_p\left(L'(1,E)/\Omega_E R_E\right) = \mathrm{ord}_p\left(\#Ш(E/\mathbb{Q}) \prod c_\ell(E)\right)$.

Thm B: Assume (H).

(a) $\#\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) < \infty \iff \begin{matrix} \mathrm{rk}\, E(\mathbb{Q}) = 0 \\ L(1,E) \neq 0 \end{matrix}$, $\#Ш(E/\mathbb{Q}) < \infty$.

(b) $\mathrm{crk}\, \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1 \iff \begin{matrix} \mathrm{rk}\, E(\mathbb{Q}) = 1 \\ \mathrm{ord}_{s=1} L(s,E) = 1 \end{matrix}$, $\#Ш(E/\mathbb{Q}) < \infty$

Thm C: ~~[illegible]~~.

(a) $\geq 20\%$ have $\mathrm{rk}(E) = \mathrm{rk}^{an}(E) = 0$

(b) $\geq 24\%$ have $\mathrm{rk}(E) = \mathrm{rk}^{an}(E) = 1$

$c(\,) \geq 79\%$. $rk = rk^{an}$ $(= 0 \text{ or } 1)$

Thms A & B are deduced using Iwasawa theory.

Thm C follows from Thm B and counting methods of Bhargava, Shankar.

Thm A & Thm B: Due to work of

- Kato
- Kobayashi
- Skinner-Urban
- Xin Wan
- Castella
- Gross-Zagier
- Kolyvagin
- ⋮
- Lei-Loeffler-Zerbes

① Kato's Work

$\Gamma = Gal(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_p$

$\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$.

$T = T_p E$

If E ordinary at p, then there is a filtration

$0 \to T^+ \to T \to T^- = T/T^+ \to 0$

$\begin{pmatrix} \alpha^{-1}\varepsilon & * \\ 0 & \alpha \end{pmatrix}$ rep of $G_{\mathbb{Q}_p}$

$H^1(\mathbb{Z}[\tfrac{1}{p}], T\otimes\Lambda) \supseteq z_{Kato}$ "zeta element"

free Λ-module of rk 1.

$$H^1(\mathbb{Q}_p, T\otimes\Lambda)/H^1(\mathbb{Q}_p, T^+\otimes\Lambda) \xrightarrow{col} \Lambda$$

$$z_{Kato} \longrightarrow L_p$$

$$\mathrm{ch}_\Lambda\left(\frac{H^1(\mathbb{Z}[\frac{1}{p}], T\otimes\Lambda)}{z_{Kato}}\right) \overset{?}{\subseteq} \mathrm{ch}_\Lambda(H^2(\mathbb{Z}_p[\tfrac{1}{p}], T\otimes\Lambda))$$

(this inclusion doesn't require ordinary assump.)

$\| \leftarrow$ via global duality.

$\mathrm{ch}_\Lambda(\mathrm{Sel}_{strict}(T\otimes\Lambda^*)^*)$ ← Pont. dual

↖ condition at p is 0.

$$0 \to H^1(\mathbb{Z}_p[\tfrac{1}{p}], T\otimes\Lambda) \to H^1(\mathbb{Q}_p, T\otimes\Lambda)/H^1(\mathbb{Q}_p, T^+\otimes\Lambda) \to (\mathrm{Sel}_{Gr}(T\otimes\Lambda^*))^*$$

$$\downarrow$$

$$(\mathrm{Sel}_{str.}(T\otimes\Lambda^*))^*$$

$$\downarrow$$

$$0$$

Gr = Greenberg.

Kato: $\mathrm{ch}_\Lambda(\mathrm{Sel}_{Gr}(T\otimes\Lambda^*)^*) \mid (L_p(E))$

(p ordinary)

S.-Urban: other direction Eisenstein congruences on $U(2,2)$ and Galois reps.

Kobayashi found right analogue of Greenberg Selmer condition in supersingular case with $\pm$ Selmer groups.

The approach using $U(2,2)$ doesn't seem to show if the classes lie in the $\pm$ spaces.

2 Main Conjectures over K

K imag. quad. $\quad$ p splits, $p = v\bar{v}$

K_∞/K max. $\mathbb{Z}_p$-ext, $\Gamma_K = \mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^2$

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$\Lambda_K = \mathbb{Z}_p[\![\Gamma_K]\!]$, $\quad \psi : G_K \twoheadrightarrow \Gamma_K \to \overline{\mathbb{Q}}_p^\times$

Hodge-Tate wts $(n, -n)$ $\quad n \geq 0$

HT wt of cycl. char $= -1$

$L(s, E, \psi^{-1})$

$s = 1$ critical value $\quad$ period $= \begin{cases} \Omega_E^+ \Omega_E^- & n = 0 \quad \text{(fin. order char)} \\ \Omega_K^{4n} & n > 0 \end{cases}$

(b/c if $n > 0$, ψ gives CM mod form of higher wt so that is the correct period)

Two p-adic L-functions:

$\Lambda_K \ni$ $L_p(E/K)$ interpolates $n = 0$

$\mathcal{L}_p(E/K)$ interpolates $n > 0$

Two main conjectures:

(ordinary case here for simplicity only)

$\mathrm{Sel}_{Gr}(T_p \otimes \Lambda_K^*)$ $\quad$ (p-ordinary)

$\mathrm{Sel}_v(T_p \otimes \Lambda_K^*)$ $\quad$ res at $\bar{v} = 0$.

$H^1_{v,f} := $

$H^1_{v,f}(\mathcal{O}_K[\tfrac{1}{p}], T \otimes \Lambda_K)$ $\quad$ res at $\bar{v}$ $\in$ im$(H^1(K_v, T^{-} \otimes \Lambda_K))$

$0 \to H^1_{v,f} \to H^1(K_v, T \otimes \Lambda_K)/H^1(K_v, T^+ \otimes \Lambda_K) \to (\mathrm{Sel}_{Gr})^* \to (\mathrm{Sel}_{str,Gr})^* \to 0$

str at v, Gr at v^*

$0 \to H^1_{v,f} \to H^1(K_{\bar{v}}, T^+ \otimes \Lambda_K) \to (\mathrm{Sel}_v)^* \to (\mathrm{Sel}_{str,Gr})^* \to 0$

One now considers the Beilinson-Flach elements and work of Lei-Loeffler-Zerbes. to get both m.c. at once. They prove the analogue of Kato's result for Beilinson-Flach elements. Xin Wan then proves the converse.

This essentially allows you to deal with supersingular case ($n=0$) by dealing with ordinary case ($n>0$).