§0 Motivation:

(work of Mazur ~77)

$X_0(N)$, N prime $/\mathbb{C}$ $\quad \underbrace{\Gamma_0(N)\backslash \mathfrak{H}}_{Y_0(N)} \cup \{\text{cusps}\}$

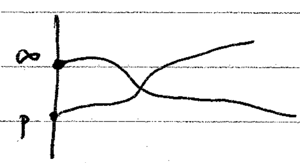
$X_0(N)/\mathbb{Q}$ and $Y_0(N)$ parameterizes elliptic curves $\{(E, H)\}$ for H a subgroup of order N.

Thm (Mazur): If $N > 163$, then $Y_0(N)(\mathbb{Q}) = \emptyset$.
($\Rightarrow \not\exists$ e.c. $E/\mathbb{Q}$ with a rational $\overset{\wedge}{\text{cyclic}}$ subgroup of order $N > 163$.)

Two ideas of Mazur:

(1) To prove $X_0(N)(\mathbb{Q})$ is finite, exhibit an optimal quotient $J_0(N) \twoheadrightarrow A$ s.t. $A(\mathbb{Q})$ is finite ($J_0(N) = \mathrm{Jac}(X_0(N))$). He used the Eisenstein quotient.
- Using Kolyvagin, it suffices to exhibit $\forall\ N > N_0$, quotient $A = A_f$, f newform s.t. $L(c, f) \neq 0$ where c is center point.

(2) $X_0(N) / \overbrace{\mathbb{Z}[\tfrac{1}{5}]}^{R}$
$\nwarrow$ arith. surface



$$\begin{tikzcd} \cot(A) \arrow[r] & \cot(J_0(N)) \arrow[r] & H^0(X_0(N), \Omega^1)_R \arrow[r, "q\text{-exp}"] \arrow[d] & R[[q]] \arrow[dl] & \sum a_n q^n \arrow[d, mapsto] \\ & & \cot(X_0(N))_R \simeq R & & a_1 \end{tikzcd}$$

Picard modular surfaces, residual Albanese quotients and rational points

Proved a formal immersion at ∞

$\Rightarrow \quad p \longleftrightarrow (E, H)$

good red at $p > p_0$

$A = A_f \quad a_{f(1)} = 1.$

§1 Picard modular surfaces

$X = X_\Gamma \supset Y_\Gamma$ $\quad\Gamma$ arithmetic subgroup of $U(2,1)$ acting on the unit ball in $\mathbb{C}^2$.

$Y_\Gamma \cup \{\text{cusps}\}$ Baily-Borel-Satake compactification.

smooth resolution (toroidal)

defined over F_Γ a number field.

Thm: (Dimitrov-R.): Y_Γ is Mordellic, for suitable finite index subgroup $\Gamma' \subset \Gamma$, i.e., has only finitely many points over any f.g. field $k \supset F_{\Gamma'}$

$\Gamma = SL_2(\mathcal{O}_M)$, $\quad M = \mathbb{Q}(\sqrt{-D})$, $\quad D > 0$

$\infty \leadsto E_\infty$ canonical elliptic curve w/ CM by $\mathcal{O}_M$

(minimal conductor) (studied by Gross)

X_Γ has h_M cusps.

Interested in $\Gamma = \Gamma_0(N)$, $\quad \{(A, \varphi, \iota: \mathcal{O}_M \hookrightarrow \operatorname{End} A, H)\}$

ab. 3-fold

rk $3/\mathcal{O}_M$

$H' \subset H'^{\perp} \subset H.$

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8-18-16
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§2 Albanese variety of residual quotients

$X = X_\Gamma$ everything defined over M

$$X \longrightarrow Alb(X) \;/\mathbb{C} \qquad H^0(X, \Omega^1)^* / H_1(X, \mathbb{Z})$$

Fourier-Jacobi exp. at ∞

$$H^0(X, \Omega^1) \ni \omega = \sum_{n \geq 1} a_n(\omega)\, q^n \frac{dq}{q} + \sum_{n \geq 0} b_n(\omega)\, q^n\, dw$$

where $a_n(\omega), b_n(\omega)$ are theta functions, $b_0(\omega) = b_0$ (constant)

$\omega \longleftrightarrow$ automorphic form f, which contributes to H^1.
f cuspidal iff $b_0 = 0$. Otherwise f is residual.

§3 Galois representations attached to $H^1(X)$

$X = X_K$, K compact open subgroup of $G(\mathbb{A}_f)$,

$G = U(M^3, \Phi)_{/\mathbb{Q}}$ hermitian form of signature $(2,1)$

$M = \mathbb{Q}(\sqrt{-D})$ as above.

$$X_K(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K \qquad K_\infty = U(2) \times U(2)$$

finite union of X_Γ's.

$\omega \in H^0(\Omega^1_X) \longleftrightarrow \pi$ auto rep. $= \pi_\infty \otimes \pi_f$ of $G(\mathbb{A}_\mathbb{Q})$

s.t. π_∞ contributes to $H^1(\mathfrak{g}_\infty, K_\infty, \mathbb{C})$
and $\pi_f^K \neq 0$.

$$\omega \rightsquigarrow \pi \longleftrightarrow \rho : \mathcal{L}_M \times SL(2) \to GL(3)$$

$\uparrow$
Langlands group $\twoheadrightarrow W_M$ - Weil group.

s.t. (1) ρ conjugate self-dual s.t. $\mathrm{Ind}_M^{\mathbb{Q}} \rho$ is orthogonal

(2) $\rho|_{W_{\mathbb{C}} = \mathbb{C}^\times} : z \longmapsto \begin{pmatrix} z^{-1} & & \\ & z/\bar{z} & \\ & & \bar{z} \end{pmatrix}$

Consider the 3-dimensional rep. of $SL(2)$: $\underset{\substack{\uparrow \\ \text{2-dim.}}}{\underline{St}} \otimes 1$.

$\mathcal{L}_M$ commutes with $SL_2 \Rightarrow \mathcal{L}_M$ acts via $\mathcal{L}_M^{ab} = W_M^{ab} \underset{CFT}{\cong} \mathbb{A}_M^\times / M^\times$.

<u>Upshot:</u> $\rho = \underline{St} \otimes \lambda \oplus 1 \otimes \chi$
with λ conjugate symplectic and χ conjugate orthogonal.

Infinity type $\Rightarrow \lambda$ is a wt 1 Hecke char. of M
and χ is a wt -2 Hecke char of M.

Put $\varphi = \lambda \chi^{-1}$ wt 3 Hecke char.

<u>Cotangent map:</u>

$\mathrm{Cot}_\infty(X_K \mathbb{P}_{/\mathbb{C}}) \simeq \mathbb{C}^2$

$$H^0(\Omega^1_{X_K}) \longrightarrow \mathrm{Cot}_\infty(X_{K/\mathbb{C}}) \simeq \mathbb{C}^2$$
$$\underset{\omega}{\Psi} \longmapsto (a_1(\omega), b_0)$$

<u>Consequence:</u> $L(H^1(\mathrm{Alb}(X_K)), s)$ is a product of Hecke L-functions

$\Rightarrow \mathrm{Alb}(X_K)$ is of CM type.

§4 Existence of Residual forms.

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))_{disc} = L^2_{cusp} \oplus \underbrace{L^2_{res.}}$$

these are exhausted by residues of E.S.

(λ, χ) data interested in

$E(y,s)$ Eis serie related to H^1. Poles are given by those of
$$\frac{L(s,\varphi)\,\zeta(2s)}{L(s+1,\varphi)\,\zeta(2s+1)}.$$

Existence $\Leftrightarrow L(\text{center}, \varphi) \neq 0$.

§5 The End

We have explicit formulae for b_0, $a_1(\omega)$ with complete knowledge of b_0 (quotient of L-values), $a_1(\omega)$ is difficult. Need $\exists\ (\lambda, \chi)$ s.t. $L(\text{center}, \varphi) \neq 0$, $L(\text{center}, \chi\cdot) \neq 0$, $a_1(\omega) \not\equiv 0 \pmod{p}$, p large.