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Beyond the main conjecture in Iwasawa Theory:

joint w/ Stone, Snellman

Fitting Filtrations:

R commutative ring with id.

M an R-module, finitely presented.

$R^m \to R^n \to M \to 0$ $m \times n$ matrix A

$Fit_R^i(M)$ = ideal in R generated by det($(n-i)\times(m-i)$ minors of A)
for $i \geq 0$.

$Fit_R^0(M) \subseteq Fit_R^1(M) \subset \cdots \subset Fit_R^n(M) = R$

Note: R is a PID and M a torsion module, $\Rightarrow Fit_R^* M$ determines M up to isom.

Iwasawa modules:

$R = \Lambda = \mathcal{O}[[t]]$ $\mathcal{O} \supseteq \mathbb{Z}_p$ finite extension.

Let X be a Λ-module. Assume X is f.g and torsion, projective dimension ≤ 1 over Λ ($pd_\Lambda X \leq 1$). , i.e. X contains no finite Λ-submodule.

Main Conj.: $Fit_\Lambda^0(X) = (char_\Lambda(X)) = ($ p-adic L-fctn $)$

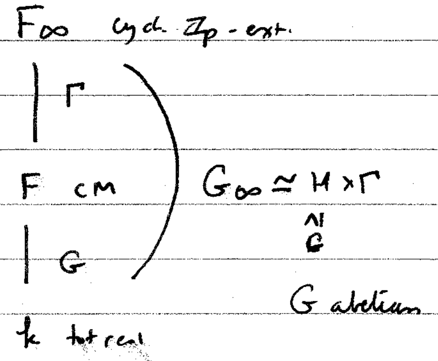
What if you knew $Fit_\Lambda^* X$?

Suppose $\mathrm{Fit}^*_\Lambda(X) = \mathrm{Fit}^*_\Lambda(X')$. This implies $X \sim X'$ (quasi-isomorphic.) However, if $X \sim X'$ does not imply $\mathrm{Fit}^*_\Lambda(X) = \mathrm{Fit}^*_\Lambda(X')$.

The statement $\mathrm{Fit}^*_\Lambda(X) = \mathrm{Fit}^*_\Lambda(X')$ does not imply $X \simeq X'$.
[Examples by Popescu, Kurihara]

Kurihara (2001) - conjectured $\mathrm{Fit}^*_\Lambda(X)$ (← classical, $k=\mathbb{Q}$) under very restrictive conditions motivated by work on Euler systems by Rubin - Kolyvagin. and an idea by Schoof (80's).

Today formulate a general conjecture for $\mathrm{Fit}^*_{\Lambda_G}(T_p(M_{S,T}))$. and what we can prove. We also show Kurihara's conjectures fit into this framework as well.



$$\Lambda_G = \mathbb{Z}_p[[G_\infty]]^-$$ ← wrt complex conj.

$$(p>2) \qquad \simeq \mathbb{Z}_p[H]^-[[t]]$$

Iwasawa modules (Greither - P.)

S, T finite sets of primes in k.

$S = S_{ram}(F_\infty/k)$
T contains 2 primes of distinct residue char.
Assume $S \cap T = \phi$.

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__abstract p-adic 1-motive__

$$\mu_{S,T} := \left[\mathcal{X}_S = [\mathrm{Div}_{F_\infty}(S \setminus S_p) \otimes \mathbb{Z}_p]^- \xrightarrow{d} J_T := \varinjlim_n (C_{F_n,T} \otimes \mathbb{Z}_p)^- \right]$$

↑ $\mathbb{Z}_p$ - free of finite rank   Assume $\mu = 0$, so ↗ div. torsion finite corank.

$\downarrow T_p$ (p-adic realization functor)

$$0 \to T_p(\mathcal{O}_T) \to T_p(\mu_{S,T}) \longrightarrow \mathcal{X}_S \to 0 \quad [\Lambda_G\text{ - modules}]$$

__Thm (Greither - P.):__

1) $\mathrm{pd}_{\Lambda_G} T_p(\mu_{S,T}) \leq 1.$

2) $\mathrm{Fitt}^0_{\Lambda_G}(T_p(\mu_{S,T})) = (\Theta^{\infty}_{S,T})$ ↖ equivariant p-adic L-fctn.

$$\Theta^{(0)}_{F/k,S,T} := \prod_{v \in T} (1 - \sigma_v^{-1} N v) \left(\sum_{\chi \in \hat{G}} L_S(0,\chi)\, e_{\chi^{-1}} \right)$$

$\in \mathbb{Z}_p[G]^-$

↖ Cassou-Noguès, Deligne-Ribet

$$\left\{ \Theta^{(0)}_{F_n/k,S,T} \right\}_n \in \varprojlim_n \mathbb{Z}_p[G(F_n/k)] = \Lambda_G.$$

$\parallel$

$\Theta^{\infty}_{S,T}$

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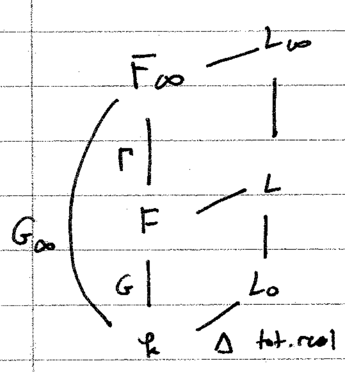
Denote $\Theta_{S,T}^{\infty}$ by Θ_F^{∞}.

<u>Question</u>: What is $\mathrm{Fit}^i_{\Lambda_G}(T_p(M_{S,T}))$?

Fix $p^N \gg 1$.



$\Delta \simeq (\mathbb{Z}/p^N\mathbb{Z})^r$, $L_0 \cap F_\infty = k$.

fix generators $\sigma_1, \dots, \sigma_r$

$$\mathbb{Z}_p[[G(L_\infty/k)]]^- \simeq \Lambda_G[\Delta] \simeq \frac{\Lambda_G[x_1, \dots, x_r]}{((x_i+1)^{p^N}-1)}$$

$$\Theta \longmapsto \sum \partial_{i_1,\dots,i_r}(\Theta)\, x_1^{i_1} \cdots x_r^{i_r}$$

<u>Lemma</u>: If $i_1, \dots, i_r \leq 1$, then $\partial_{i_1,\dots,i_r}(\Theta)$ are unique in Λ_{G/p^N}.

<u>Conjecture 1</u>: F/k, S,T as above. For all i

$$F_N^i \quad \mathrm{Fit}^i_{\Lambda_{G/p^N}}(T_p(M_{S,T})/p^N) = \langle \Theta_F^{\infty}, \partial_{i_1,\dots,i_r}(\Theta_L) \mid \forall r \geq 0,\ \forall L_0 \text{ as above},\ \forall i_1,\dots,i_r \leq 1,\ i_1+\dots+i_r \leq i \rangle = \mathcal{F}_N^i$$

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Thm: If $k = \mathbb{Q}$, then Conjecture 1 holds.

Comment: ${}^{\sigma}\mathcal{F}_N^i \subseteq F_N^i$ can be shown over any base field basically via Schaaf & codescent data for $T_p(M_{S,T})$.

$F_N^i \subseteq {}^{\sigma}\mathcal{F}_N^i$ for $i \geq 1$ is hard. The Euler system technique breaks down here. Here use Gross's p-adic refinement of the Rubin-Stark conjecture.

Back to Kurihara:

$$X_T := \varprojlim \left(C_{F_n,T} \otimes \mathbb{Z}_p \right)^-.$$

$G \xhookrightarrow{\chi} \mathbb{C}^\times$ of order coprime to p. (**)

$k = \mathbb{Q}$, p does not split in F/k. (*)

He conjectures
$$\mathrm{Fit}^i_{\Lambda_G^\chi/p^N}\left(X_T^\chi\right) = \left\{ {}^{\sigma}\mathcal{F}_N^i \right\}^\chi \quad \forall i.$$

(**) $\Rightarrow \mathcal{Z}_S^\chi = 0$.

(*) $\Rightarrow T_p(J_T)^\chi \simeq X_T^\chi$.

Thm: Kurihara's conjecture is true.